SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

THE FIELD OF MULTISYMMETRIC FUNCTIONS

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Take n independent transcendentals $x_1, \cdots, x_n$ over a field k, and let the symmetric group G on n letters act as usual on $k(x_1, \cdots, x_n)$. The invariant subfield is again purely transcendental, because classical algebra says that it is generated by the n elementary symmetric functions.

Take nr independent transcendentals, in n sets of r each:

$$\{x_{1\alpha}\}, \{x_{2\alpha}\}, \cdots, \{x_{n\alpha}\}, \qquad \alpha = 1, \cdots, r,$$

and denote by $k(x)_r$ the field they generate over k. The same symmetric group G now acts on $k(x)_r$ by permuting the n sets among each other; that is, $g(x_{i\alpha}) = x_{g(i)\alpha}$. The G-invariant elements are called multisymmetric functions, and again it is a classical result that the field $k(x)_r^G$ they form can be generated by the elementary multisymmetric functions. (These are the coefficients of $\prod(1+x_{i1}U_1+ \cdots +x_{ir}U_r)$ when written out as a polynomial in the indeterminates U_α.) But if n, $r>1$, these elementary functions cannot be independent because there are more than nr of them. Nevertheless,[2]

THEOREM. *The field $k(x)_r^G$ is a purely transcendental extension of k.*

In the language of algebraic geometry, *the n-fold symmetric product of a rational variety is again rational.* Our proof, like most algebraic arguments, succeeds in both simplifying and effectively concealing the original geometric ideas which produced it.

PROOF. We know it when $r=1$, so we assume $r>1$ and use induction. From the field $k(x_{11}, \cdots, x_{1,r-1})$ select any n elements $a_1^{(1)}, \cdots, a_1^{(n)}$ linearly independent over k, and denote by $a_i^{(1)}, \cdots, a_i^{(n)}$

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[2] I am informed that G. Bergman has a proof in characteristic zero (unpublished), based on formulas from the representation theory for the symmetric group. Note also that a multisymmetric polynomial can be expressed as a polynomial in the elementary multisymmetric functions if the characteristic is 0, but in characteristic p this is not always so. It is true when $r=1$; for $r>1$, see Nagata, Mem. Coll. Sci. Univ. Kyoto 29 (1955), 165.

the corresponding elements in $k(x_{i1}, \cdots, x_{i,r-1})$, i.e., the images under the action of G. Then we denote by $t_1, \cdots, t_n$ the unique elements of $k(x)_r$ satisfying the n linear inhomogeneous equations

$$a_i^{(1)} t_1 + \cdots + a_i^{(n)} t_n = x_{ir}, \qquad i = 1, \cdots, n. \tag{1}$$

By induction on r, we will prove the theorem if we can show that

$$k(x)_r^G = k(x)_{r-1}^G(t_1, \cdots, t_n). \tag{2}$$

First of all, the inclusion $\supseteq$ holds, since the t_i are G-invariant (because the system of equations is—or use Cramer's rule). So it is enough to show that $k(x)_r$ has degree $n!$ over both fields in (2). For the left-hand field, this is clear by Galois theory. As for the right, we remark that the system (1) shows that

$$k(x)_r = k(x)_{r-1}(t_1, \cdots, t_n).$$

Thus the t_i are independent transcendentals over $k(x)_{r-1}$ and hence

$$\begin{aligned}[k(x)_r : k(x)_{r-1}^G(t)] &= [k(x)_{r-1}(t) : k(x)_{r-1}^G(t)] \\ &= [k(x)_{r-1} : k(x)_{r-1}^G] = n!\end{aligned}$$

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