A GENERALIZED TWO-POINT BOUNDARY VALUE PROBLEM

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1. **Introduction.** The purpose of this paper is to prove there exists a unique solution of the two-point boundary value problem

(1.1) $x''=f(t, x, x')$,

(1.2) $a_0x(a)-a_1x'(a)=\alpha$,

(1.3) $b_0x(b)+b_1x'(b)=\beta$

assuming that:

(i) $f(t, x, x')$ is a continuous real valued function defined on $S\equiv\{(t, x, x')\mid a\leqq t\leqq b, |x|+|x'|<\infty\}$,

(ii) $f(t, x, x')$ is nondecreasing on S with respect to x,

(iii) $|f(t, x, x_1')-f(t, x, x_2')|\leqq M|x_1'-x_2'|$ on S,

and in addition assuming that a_0, a_1, b_0, $b_1\geqq 0$, $a_0+b_0>0$, $a_0+a_1>0$, and $b_0+b_1>0$.

By boundary value problem (BVP) (1.1), we shall mean equation (1.1) together with boundary conditions (1.2) and (1.3). BVP(1.1) was investigated in [**5**]. In this paper, the assumptions on $f(t, x, x')$ are weakened considerably and a much simpler proof of the existence of a unique solution to BVP(1.1) is given.

The main result of this note is Theorem 3.1. The proof makes use of the existence and uniqueness theory developed using the subfunction approach in [**1**] and [**2**]. In particular, under conditions (i), (ii), and (iii), the two-point boundary value problem: $x''=f(t, x, x')$, $x(a)=\gamma$, $x(b)=\delta$, has a unique solution $u(t, \gamma, \delta)$ which depends continuously on the boundary data (γ, δ).

2. **Preliminary results.** In this section, we prove a sequence of lemmas.

LEMMA 2.1. *Let $\phi(s)$ be a positive continuous function such that*

$$\int^{\infty}\frac{sds}{\phi(s)}=\infty. \tag{2.1}$$

Given $R>0$, there exists $M_R>0$ such that if $x(t)\in C^2[a, b]$, satisfies $|x(t)|\leqq R$ and $|x''(t)|\leqq\phi(|x'(t)|)$ on $[a, b]$, then $|x'(t)|\leqq M_R$. Moreover, $M_R\to 0$ as $R\to 0$.

PROOF. Given $R>0$, by (2.1) there exists $M_R>0$ such that

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$$\int_{2R/(b-a)}^{M_R} \frac{sds}{\phi(s)} = 2R. \tag{2.2}$$

Then $|x'(t)| \leqq M_R$ follows (Hartman [**3**, pp. 428–429]). From (2.2), we see that if $R \to 0$, then $M_R \to 0$.

LEMMA 2.2. *If* $f(t, x, x')$ *satisfies* (i), (ii), (iii), *then the boundary value problem*

$$x'' = f(t, x, x') \qquad x(c) = \gamma, \quad x(d) = \delta \tag{2.3}$$

has a unique solution $u(t, \gamma, \delta)$ *for any* $a \leqq c \leqq d \leqq b$.

PROOF. This is a corollary to [**1**, Theorem 6.2, p. 1064].

LEMMA 2.3. *If* $f(t, x, x')$ *satisfies* (i), (ii), (iii), *then* $u(t, \bar{\gamma}, \bar{\delta}) \to u(t, \gamma, \delta)$ *and* $u'(t, \bar{\gamma}, \bar{\delta}) \to u'(t, \gamma, \delta)$ *uniformly on* $[c, d]$ *as* $|\gamma - \bar{\gamma}| + |\delta - \bar{\delta}| \to 0$.

PROOF. Given $\epsilon > 0$, by Jackson and Fountain [**2**, Theorem 9, p. 1262], $u(t, \gamma, \delta) + \epsilon$ is a superfunction and $u(t, \gamma, \delta) - \epsilon$ is a subfunction relative to solutions of $x'' = f(t, x, x')$. Hence, if $|\bar{\delta} - \delta| + |\bar{\gamma} - \gamma| < \epsilon$, then $u(t, \gamma, \delta) - \epsilon \leqq u(t, \bar{\gamma}, \bar{\delta}) \leqq u(t, \gamma, \delta) + \epsilon$ for all $t \in [c, d]$; and $u(t, \bar{\gamma}, \bar{\delta}) \to u(t, \gamma, \delta)$ uniformly on $[c, d]$ as $|\bar{\delta} - \delta| + |\bar{\gamma} - \gamma| \to 0$.

We have left to prove that $u'(t, \bar{\gamma}, \bar{\delta}) \to u'(t, \gamma, \delta)$ as $|\bar{\gamma} - \gamma| + |\bar{\delta} - \delta| \to 0$.

Define $K \equiv \{(t, y) \mid c \leqq t \leqq d,\ u(t, \gamma - 1, \delta - 1) \leqq y \leqq u(t, \gamma + 1, \delta + 1)\}$ and $\phi(s) \equiv Ms + \max_K |f(t, y, 0)|$. Note that $\int^\infty (sds/\phi(s)) = \infty$. For $|\gamma - \bar{\gamma}| \leqq 1$ and $|\delta - \bar{\delta}| \leqq 1$, we have

$$\begin{aligned} |u''(t, \bar{\gamma}, \delta)| &= |f(t, u(t, \bar{\gamma}, \bar{\delta}), u'(t, \bar{\gamma}, \bar{\delta}))| \\ &\leqq M|u'(t, \bar{\gamma}, \bar{\delta})| + |f(t, u(t, \bar{\gamma}, \bar{\delta}), 0)| \\ &\leqq \phi(|u'(t, \bar{\gamma}, \bar{\delta})|) \end{aligned}$$

and

$$\begin{aligned} |u(t, \bar{\gamma}, \bar{\delta})| &\leqq B \\ &\equiv \max_{[a,b]} \{\max[|u(t, \gamma - 1, \delta - 1)|, |u(t, \gamma + 1, \delta + 1)|]\}. \end{aligned}$$

Hence, by Lemma 2.1, there exists $N_B > 0$ such that $|u'(t, \bar{\gamma}, \bar{\delta})| \leqq N_B$ for $|\gamma - \bar{\gamma}| \leqq 1$ and $|\delta - \bar{\delta}| \leqq 1$.

Define $\phi_1(s) \equiv 2 \max |f(t, y, y')|$ over $\{(t, y, y') \mid (t, y) \in K, |y'| \leqq N_B\}$ and $R \equiv \max |u(t, \bar{\gamma}, \bar{\delta}) - u(t, \gamma, \delta)|$ on $[c, d]$. Note that $\int^\infty (sds/\phi_1(s)) = \infty$. For $|\gamma - \bar{\gamma}| \leqq 1$ and $|\delta - \bar{\delta}| \leqq 1$, we have

$$
\begin{aligned}
|u''(t,\bar\gamma,\bar\delta) - u''(t,\gamma,\delta)| &\leqq |f(t,u(t,\bar\gamma,\bar\delta),u'(t,\bar\gamma,\bar\delta))| \\
&\quad + |f(t,u(t,\gamma,\delta),u'(t,\gamma,\delta))| \\
&\leqq \phi_1(|u'(t,\bar\gamma,\bar\delta) - u'(t,\gamma,\delta)|).
\end{aligned}
$$

By Lemma 2.1, $|u'(t,\bar\gamma,\bar\delta)-u'(t,\gamma,\delta)|\leqq M_R$ where $M_R\to 0$ as $R\to 0$. Since $R\to 0$ as $|\gamma-\bar\gamma|+|\delta-\bar\delta|\to 0$, we conclude that $u'(t,\bar\gamma,\bar\delta)\to u'(t,\gamma,\delta)$ uniformly on $[c,d]$ as $|\gamma-\bar\gamma|+|\delta-\bar\delta|\to 0$.

LEMMA 2.4. *If $f(t,x,x')$ satisfies* (i), (ii), *and* (iii), $\gamma_2\geqq\gamma_1$ *and* $\delta_2\geqq\delta_1$, *then* $z(t)\equiv u(t,\gamma_2,\delta_2)-u(t,\gamma_1,\delta_1)$ *is a subfunction with respect to solutions of*

$$u'' = -M|u'| \tag{2.4}$$

on $[c,d]$. *If* $\delta_2-\delta_1\geqq\gamma_2-\gamma_1$, *then*

$$
\begin{aligned}
0 &\leqq z(t) \leqq (\gamma_2-\gamma_1) + [(\delta_2-\delta_1)-(\gamma_2-\gamma_1)] \\
&\quad\cdot\left[\int_c^t \exp(-M(s-c))ds\right]\Big/\left[\int_c^d \exp(-M(s-c))ds\right] \\
&\equiv \omega_1(t).
\end{aligned} \tag{2.5}
$$

If $\delta_2-\delta_1\leqq\gamma_2-\gamma_1$, *then*

$$
\begin{aligned}
0 &\leqq z(t) \leqq (\gamma_2-\gamma_1) + [(\delta_2-\delta_1)-(\gamma_2-\gamma_1)] \\
&\quad\cdot\left[\int_c^t \exp(M(s-c))ds\right]\Big/\left[\int_c^d \exp(M(s-c))ds\right] \\
&\equiv \omega_2(t).
\end{aligned} \tag{2.6}
$$

PROOF. By the uniqueness of solutions to the two-point boundary value problem in Lemma 2.2, we must have $u(t,\gamma_2,\delta_2)\geqq u(t,\gamma_1,\delta_1)$. By properties (ii) and (iii),

$$
\begin{aligned}
z''(t) &= f(t,u(t,\gamma_2,\delta_2),u'(t,\gamma_2,\delta_2)) - f(t,u(t,\gamma_1,\delta_1),u'(t,\gamma_1,\delta_1)) \\
&\geqq f(t,u(t,\gamma_1,\delta_1),u'(t,\gamma_2,\delta_2)) - f(t,u(t,\gamma_1,\delta_1),u'(t,\gamma_1,\delta_1)) \\
&\geqq -M|u'(t,\gamma_2,\delta_2)-u'(t,\gamma_1,\delta_1)| = -M|z'(t)|.
\end{aligned}
$$

By Jackson and Fountain [**2**, Theorem 8, p. 1261], this inequality implies that $z(t)$ is a subfunction with respect to (2.4) on $[c,d]$.

If $\delta_2-\delta_1\geqq\gamma_2-\gamma_1$, then $\omega_1(t)$ is a solution to (2.4) satisfying $\omega_1(c)=z(c)$ and $\omega_1(d)=z(d)$. Similarly, if $\gamma_2-\gamma_1\leqq\delta_2-\delta_1$, then $\omega_2(t)$ is such a solution. Then inequalities (2.5) and (2.6) follow from the definition of subfunction (see Jackson and Fountain [**2**, p. 1254]).

LEMMA 2.5. *If $f(t,x,x')$ satisfies* (i), (ii), *and* (iii) *and if* $b_0, b_1\geqq 0$

and $b_0+b_1>0$, *then the boundary value problem*

$$x'' = f(t, x, x') \qquad x(a) = s, \quad b_0x(b) + b_1x'(b) = \beta \tag{2.7}$$

has a unique solution $u(t, s, \delta(s))$ *on* $[a, b]$ *where* $u(b, s, \delta(s)) = \delta(s)$.

PROOF. By Lemma 2.2, the problem

$$x'' = f(t, x, x') \qquad x(a) = s, \quad x(b) = \delta$$

has a unique solution $u(t, s, \delta)$ on $[a, b]$.

Define $F_s(\delta) \equiv b_0\delta + b_1u'(b, s, \delta)$. It is sufficient to show that there exists a unique $\delta(s)$ such that $F_s(\delta(s)) = \beta$.

From Lemma 2.3 we have that $u'(b, s, \delta)$ is a continuous function of δ on $(-\infty, +\infty)$ for fixed s. Hence, $F_s(\delta)$ is continuous on $(-\infty, \infty)$. Suppose $\delta_2 \geqq \delta_1$. We have

$$F_s(\delta_2) - F_s(\delta_1) = b_0(\delta_2 - \delta_1) + b_1(u'(b, s, \delta_2) - u'(b, s, \delta_1)).$$

Applying Lemma 2.4 (with $\gamma_1 = \gamma_2 = s$), since $z(b) = u(b, s, \delta_2) - u(b, s, \delta_1) = \omega_1(b)$, we have

$$\begin{aligned} z'(b) &= u'(b, s, \delta_2) - u'(b, s, \delta_1) \geqq \omega_1'(b) \\ &= (\delta_2 - \delta_1) \exp[-M(b - a)] \Big/ \int_a^b \exp[-M(s - a)]ds. \end{aligned} \tag{2.8}$$

Define $P_1 \equiv b_0 + b_1 \exp[-M(b-a)]/\int_a^b \exp[-M(s-a)]ds > 0$. Then,

$$F_s(\delta_2) - F_s(\delta_1) \geqq P_1(\delta_2 - \delta_1).$$

This inequality implies that $F_s(\delta)$ is strictly increasing and has $(-\infty, +\infty)$ for its range. Hence, there exists a unique $\delta(s)$ such that $F_s(\delta(s)) = \beta$, and $u(t, s, \delta(s))$ is the unique solution to (2.7).

LEMMA 2.6. *If* $f(t, x, x')$ *satisfies* (i), (ii), *and* (iii) *and if* $b_0, b_1 \geqq 0$ *and* $b_0+b_1>0$, *then* $u(t, \bar{s}, \delta(\bar{s})) \to u(t, s, \delta(s))$ *and* $u'(t, \bar{s}, \delta(\bar{s})) \to u'(t, s, \delta(s))$ *uniformly on* $[a, b]$ *as* $\bar{s} \to s$. *Moreover, if* $s_2 \geqq s_1$, *then*

$$0 \leqq u(t, s_2, \delta(s_2)) - u(t, s_1, \delta(s_1)) \leqq s_2 - s_1. \tag{2.9}$$

PROOF. We first establish that (2.9) holds at b, i.e. $0 \leqq \delta(s_2) - \delta(s_1) \leqq s_2 - s_1$.

If $b_1 = 0$, then $\delta(s_2) - \delta(s_1) = \beta/b_0 - \beta/b_0 = 0$.

Suppose $b_1 > 0$. Since $u(t, s, \delta(s))$ is a solution to (2.7) for every s,

$$u'(b, s_2, \delta(s_2)) - u'(b, s_1, \delta(s_1)) = -(b_0/b_1)(\delta(s_2) - \delta(s_1)). \tag{2.10}$$

Suppose $\delta(s_2) - \delta(s_1) < 0$. Then (2.10) implies that

$$u'(b, s_2, \delta(s_2)) - u'(b, s_1, \delta(s_1)) \geqq 0. \tag{2.11}$$

Define $t_0 \equiv \sup\{t \mid t \in [a, b] \text{ and } u(t, s_2, \delta(s_2)) = u(t, s_1, \delta(s_1))\}$. Note that $t_0 \in [a, b)$. Applying Lemma 2.4 (with $\gamma_1 = \gamma_2 = u(t_0, s_2, \delta(s_2))$, $\delta_2 = \delta(s_1)$, $\delta_1 = \delta(s_2)$, $[c, d] = [t_0, b]$), since $\delta(s_1) > \delta(s_2)$ and $z(b) = \delta(s_1) - \delta(s_2) = \omega_1(b)$, we obtain

$$\begin{aligned} z'(b) &= u'(b, s_1, \delta(s_1)) - u'(b, s_2, \delta(s_2)) \geqq \omega_1'(b) \\ &= (\delta(s_1) - \delta(s_2)) \\ &\quad \cdot \exp(-M(b - t_0)) \Big/ \int_{t_0}^{b} \exp(-M(s - a))ds > 0. \end{aligned} \tag{2.12}$$

But (2.11) and (2.12) yield a contradiction. Hence, $\delta(s_2) - \delta(s_1) \geqq 0$.

Suppose $\delta(s_2) - \delta(s_1) > s_2 - s_1$. Applying Lemma 2.4 (with $\gamma_1 = s_1$, $\gamma_2 = s_2$, $\delta_1 = \delta(s_1)$, and $\delta_2 = \delta(s_2)$), since $z(b) = \delta(s_2) - \delta(s_1) = \omega_1(b)$, we obtain

$$\begin{aligned} z'(b) &= u'(b, s_2, \delta(s_2)) - u'(b, s_1, \delta(s_1)) \\ &\geqq \omega_1'(b) = \frac{[(\delta(s_2) - \delta(s_1)) - (s_2 - s_1)] \exp(-M(b - a))}{\int_a^b \exp(-M(s - a))ds} > 0. \end{aligned} \tag{2.13}$$

But, since $b_0 \geqq 0$ and $b_1 > 0$, this contradicts (2.10). Hence $\delta(s_2) - \delta(s_1) \leqq s_2 - s_1$.

Now (2.6) of Lemma 2.4 yields (2.9) since $0 \leqq z(t) = u(t, s_2, \delta(s_2)) - u(t, s_1, \delta(s_1)) \leqq \omega_2(t) \leqq (s_2 - s_1)$. Moreover, $|\delta(s_2) - \delta(s_1)| \leqq |s_2 - s_1|$, i.e. $\delta(s)$ is continuous. From Lemma 2.3 we conclude that $u(t, \bar{s}, \delta(\bar{s})) \to u(t, s, \delta(s))$ and $u'(t, \bar{s}, \delta(\bar{s})) \to u'(t, s, \delta(s))$ uniformly on $[a, b]$ as $\bar{s} \to s$.

LEMMA 2.7. *If $f(t, x, x')$ satisfies* (i), (ii), (iii) *and if $a_0, a_1 \geqq 0$ and $a_0 + a_1 > 0$, then the boundary value problem*

$$x'' = f(t, x, x'), \qquad x(b) = s, \quad a_0 x(a) - a_1 x'(a) = \alpha \tag{2.14}$$

has a unique solution $u(t, \gamma(s), s)$ on $[a, b]$ where $u(a, \gamma(s), s) = \gamma(s)$. As $\bar{s} \to s$, $u(t, \gamma(\bar{s}), \bar{s}) \to u(t, \gamma(s), s)$ and $u'(t, \gamma(\bar{s}), \bar{s}) \to u'(t, \gamma(s), s)$ uniformly on $[a, b]$. Moreover, if $s_2 \geqq s_1$, then

$$0 \leqq u(t, \gamma(s_2), s_2) - u(t, \gamma(s_1), s_1) \leqq s_2 - s_1. \tag{2.15}$$

PROOF. Similar to Lemmas 2.5 and 2.6.

3. **Main result.** With the aid of the preceding lemmas we can now prove the following which is a generalization of Keller's Theorem [**4**, p. 728].

THEOREM 3.1. *If $f(t, x, x')$ satisfies* (i), (ii), *and* (iii) *and if*

$$(3.1) \qquad a_0, a_1, b_0, b_1 \geqq 0,$$

$$(3.2) \qquad a_0 + b_0 > 0, \qquad a_0 + a_1 > 0, \qquad b_0 + b_1 > 0,$$

then the boundary value problem (1.1) *has a unique solution* $u(t)$ *for any* α *and* β.

PROOF. By Lemma 2.5, the problem (2.7) has a unique solution $u(t, s, \delta(s))$ for each s. Define $G(s) \equiv a_0 s - a_1 u'(a, s, \delta(s))$. It is sufficient to show there exists a unique s such that $G(s) = \alpha$. Assume $a_0 > 0$.

By Lemma 2.6, $u'(a, s, \delta(s))$ is a continuous function of s on $(-\infty, \infty)$. Hence, $G(s)$ is continuous on $(-\infty, \infty)$.

Suppose $s_2 \geqq s_1$. We have

$$(3.3) \quad G(s_2) - G(s_1) = a_0(s_2 - s_1) - a_1(u'(a, s_2, \delta(s_2)) - u'(a, s_1, \delta(s_1))).$$

By Lemma 2.6, $0 \leqq u(t, s_2, \delta(s_2)) - u(t, s_1, \delta(s_1)) \leqq s_2 - s_1$. Moreover, $u(a, s_2, \delta(s_2)) - u(a, s_1, \delta(s_1)) = s_2 - s_1$. Hence, $u'(a, s_2, \delta(s_2)) - u'(a, s_1, \delta(s_1)) \leqq 0$. We may conclude from (3.3) that

$$G(s_2) - G(s_1) \geqq a_0(s_2 - s_1).$$

This inequality implies that $G(s)$ is strictly increasing and has $(-\infty, +\infty)$ as its range. Hence, there exists a unique s such that $G(s) = \alpha$. Thus, BVP(1.1) has a unique solution.

If $a_0 = 0$, then we must have $b_0 > 0$, and the proof may be carried out as above by reversing the roles of a, b and using Lemma 2.7.

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