ON THE HYPERFINITE II_1-FACTOR

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1. **Introduction.** In this paper, we shall consider the following problem which has been asked by several mathematicians: Can we embed any arbitrary finite factor on a separable Hilbert space into the hyperfinite II_1-factor? The answer is negative.

2. **Theorem.** First of all, we shall show,

LEMMA 1. *Let M be a finite W^*-algebra on a Hilbert space $\mathfrak{H}$ such that M' has the property P in the sense of Schwartz* [2], *$B(\mathfrak{H})$ the W^*-algebra of all bounded operators on $\mathfrak{H}$, and let N be a W^*-subalgebra of M, then there exists a linear mapping P on $B(\mathfrak{H})$ to N satisfying the following conditions*:

(1) $P(x^*)=P(x)^*$ *for* $x\in B(\mathfrak{H})$,

(2) $P(h)\geqq 0$ *for* $h(\geqq 0)\in B(\mathfrak{H})$, *and* $P(I)=I$,

(3) $P(axb)=aP(x)b$ *for* $a, b\in N$ *and* $x\in B(\mathfrak{H})$.

PROOF. By the result of Schwartz [2], there is a linear mapping P_1 on $B(\mathfrak{H})$ to M satisfying the same properties. On the other hand, by the result of Umegaki [3], there is a linear mapping P_2 on M to N satisfying the same properties. Now put $P(x)=P_2(P_1(x))$ for $x\in B(\mathfrak{H})$, then clearly P satisfies the required properties. This completes the proof.

Now let M be the hyperfinite II_1-factor, and let N be the II_1-factor generated by the left regular representation of a countable discrete free group G with two generators.

Then we shall show the following theorem.

THEOREM. *There is no subfactor in M which is *-isomorphic to N.*

To prove the theorem, we shall proceed as follows.

Now suppose that there is a subfactor N_1 of M which is *-isomorphic to N.

Let $\{\pi, \mathfrak{H}\}$ be the standard *-representation of M on a Hilbert space $\mathfrak{H}$.

Let K be an $\aleph_0$-dimensional Hilbert space, and let $\mathfrak{H}\otimes K$ be the tensor product of $\mathfrak{H}$ and K. We shall consider the W^*-representation $\{\pi\otimes I_K, \mathfrak{H}\otimes K\}$ of M on $\mathfrak{H}\otimes K$, where $\pi\otimes I_K$ is the amplification of π (cf. [1]) and I_K is the identity operator on K.

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LEMMA 2. *The commutant* $\{\pi(M)\otimes I_K\}'=\pi(M)'\otimes B(K)$ *in* $\mathfrak{H}\otimes K$ *has the property P, where* $B(K)$ *is the* W^**-algebra of all bounded operators on* K.

PROOF. $\pi(M)'$ has the property P (cf. [2]). Let a be an arbitrary element of the W^*-algebra $B(\mathfrak{H}\otimes K)$ of all bounded operators on the Hilbert space $\mathfrak{H}\otimes K$, and let $C_1(a)$ be the σ-weakly closed convex envelope of $\{u^*au \mid u \text{ (unitary)}\in I_{\mathfrak{h}}\otimes B(K)\}$, where $I_{\mathfrak{h}}$ is the identity operator on $\mathfrak{H}$. Then $C_1(a)\cap(1_{\mathfrak{h}}\otimes B(K))'=C_1(a)\cap(B(\mathfrak{H})\otimes I_K)\neq(\varnothing)$, because $I_{\mathfrak{h}}\otimes B(K)$ is a type I factor so that there exists an increasing sequence of $2^n\times 2^n$ matrix algebras $\{M_n \mid n=1, 2, 3, \cdots\}$ such that the σ-weak closure of $\bigcup_{n=1}^{\infty} M_n=I_{\mathfrak{h}}\otimes B(K)$.

Next, take $a_0\in C_1(a)\cap(B(\mathfrak{H})\otimes I_K)$, and let $C_2(a_0)$ be the σ-weakly closed convex envelope of $\{u^*a_0u \mid u \text{ (unitary)}\in\pi(M)'\otimes I_K\}$. Then clearly $C_2(a_0)\cap\{\pi(M)'\otimes I_K\}'\cap B(\mathfrak{H})\otimes I_K\neq(\varnothing)$, because $\pi(M)'$ has the property P. On the other hand, $C_2(a_0)$ is contained in the σ-weakly closed convex envelope of $\{u^*au \mid u \text{ (unitary)}\in\pi(M)'\otimes B(K)\}$. Hence $\pi(M)'\otimes B(K)$ has the property. This completes the proof.

The commutant $\{\pi(N_1)\otimes I_K\}'$ of $\pi(N_1)\otimes I_K=\pi(N_1)'\otimes B(K)$; hence it is a II_∞-factor.

Let ξ be a trace vector for M in $\mathfrak{H}$ and η be a nonzero vector in K.

Let $[\pi(N_1)\otimes I_K\xi\otimes\eta]$ be the closed subspace of $\mathfrak{H}\otimes K$ generated by $\pi(N_1)\otimes I_K\xi\otimes\eta$ and E' be the orthogonal projection of $\mathfrak{H}\otimes K$ onto $[\pi(N_1)\otimes I_K\xi\otimes\eta]$. Then the representation $x\rightarrow(\pi(x)\otimes I_K)E'(x\in N_1)$ of N_1 is standard; hence E' is a finite projection in $\{\pi(N_1)\otimes I_K\}'$.

Let $\{E_n' \mid n=1, 2, \cdots\}$ be a sequence of mutually orthogonal, equivalent projections in $\{\pi(N_1)\otimes I_K\}'$ such that $E_n'\sim E'$ and $\sum_{n=1}^{\infty} E_n'=I_{\mathfrak{h}\otimes K}$ where $1_{\mathfrak{h}\otimes K}$ is the identity operator on $\mathfrak{h}\otimes K$.

Now let G be the countable discrete free group with two generators, $L^2(G)$ the Hilbert space of all square integrable functions on G with respect to the Haar measure, $L^\infty(G)$ the algebra of all bounded functions on G.

Let $\{U, L^2(G)\}$ be the left regular representation of G on $L^2(G)$, and N be the W^*-algebra generated by $\{U(g) \mid g\in G\}$.

For $f\in L^\infty(G)$, we shall define a bounded operator T_f on $L^2(G)$ as follows:

$$T_f h = f\circ h \qquad \text{for } h\in L^2(G).$$

Let $A=\{T_f \mid f\in L^\infty(G)\}$, then A is a commutative W^*-algebra on $L^2(G)$.

N on $L^2(G)$ is standard; hence N on $L^2(G)$ is spatial isomorphic to $\{\pi(N_1)\otimes I_K\}E'$ on $E'\mathfrak{H}\otimes K$.

Since $E_n' \sim E'$ and $\sum_{n=1}^{\infty} E_n' = I_{\mathfrak{H}\otimes K}$, we can express the W^*-algebra $\{\pi(N_1)\otimes I_K\}$ as follows:

$$\{\pi(N_1) \otimes I_K\} = \mathfrak{A} \otimes I_{K_1},$$

where $\mathfrak{A}$ is a W^*-algebra on a some Hilbert space $\mathfrak{H}_1$, and $\mathfrak{A}$ on $\mathfrak{H}_1$ is spatial isomorphic to N on $L^2(G)$, and K_1 is a Hilbert space. We shall identify $\mathfrak{A}$ with N by a spatial isomorphism.

Now we shall prove the theorem.

Proof of Theorem. Now by Lemma 1, there exists a linear mapping P on $B(\mathfrak{H}\otimes K)$ to $\pi(N_1)\otimes I_K$ satisfying the properties (1)–(3).

Let τ be the trace on $\pi(N_1)\otimes I_K$.

For $f\in L^\infty(G)$, put $f_s(t)=f(s^{-1}t)$ for s, $t\in G$, then $T_{f_s}h(t)=f_s(t)h(t)=f(s^{-1}t)h(t)=U(s)T_fU(s^{-1})h(t)$ for $h\in L^2(G)$. Now we shall define a linear functional ϕ on $L^\infty(G)$ as follows: $\phi(f)=\tau(P(T_f\otimes I_{K_1}))$ for $f\in L^\infty(G)$. Then ϕ is a positive linear functional such that $\phi(1)=1$.

Moreover,

$$\begin{aligned}
\varphi(f_s) &= \tau(P(U(s)T_fU(s^{-1}) \otimes I_{K_1})) \\
&= \tau(P(U(s) \otimes I_{K_1}T_f \otimes I_{K_1}U(s^{-1}) \otimes I_{K_1})) \\
&= \tau(U(s) \otimes I_{K_1}P(T_f \otimes I_{K_1})U(s^{-1}) \otimes I_{K_1}) \\
&= \tau(P(T_f \otimes I_{K_1})) \\
&= \varphi(f) \qquad \text{for } f\in L^\infty(G).
\end{aligned}$$

Hence, ϕ will define a left invariant finitely additive measure on G. This is a contradiction (cf. [2]) and completes the proof.

Now, the following problem would be very interesting.

Problem. Can we conclude that any II_1-subfactor of the hyperfinite II_1-factor is hyperfinite?

References

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