ON THE HYPERFINITE II₁-FACTOR
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1. Introduction. In this paper, we shall consider the following problem which has been asked by several mathematicians: Can we embed any arbitrary finite factor on a separable Hilbert space into the hyperfinite II₁-factor? The answer is negative.

2. Theorem. First of all, we shall show,

Lemma 1. Let $M$ be a finite $W^*$-algebra on a Hilbert space $\mathcal{H}$ such that $M'$ has the property $P$ in the sense of Schwartz [1], $B(\mathcal{H})$ the $W^*$-algebra of all bounded operators on $\mathcal{H}$, and let $N$ be a $W^*$-subalgebra of $M$, then there exists a linear mapping $P$ on $B(\mathcal{H})$ to $N$ satisfying the following conditions:

1. $P(x^*) = P(x)^*$ for $x \in B(\mathcal{H})$,
2. $P(h) \geq 0$ for $h(\geq 0) \in B(\mathcal{H})$, and $P(I) = I$,
3. $P(axb) = aP(x)b$ for $a, b \in N$ and $x \in B(\mathcal{H})$.

Proof. By the result of Schwartz [2], there is a linear mapping $P_1$ on $B(\mathcal{H})$ to $M$ satisfying the same properties. On the other hand, by the result of Umegaki [3], there is a linear mapping $P_2$ on $M$ to $N$ satisfying the same properties. Now put $P(x) = P_2(P_1(x))$ for $x \in B(\mathcal{H})$, then clearly $P$ satisfies the required properties. This completes the proof.

Now let $M$ be the hyperfinite II₁-factor, and let $N$ be the II₁-factor generated by the left regular representation of a countable discrete free group $G$ with two generators.

Then we shall show the following theorem.

Theorem. There is no subfactor in $M$ which is *-isomorphic to $N$.

To prove the theorem, we shall proceed as follows.

Now suppose that there is a subfactor $N_1$ of $M$ which is *-isomorphic to $N$.

Let $\{\pi, \mathcal{H}\}$ be the standard *-representation of $M$ on a Hilbert space $\mathcal{H}$.

Let $K$ be an $\aleph_0$-dimensional Hilbert space, and let $\mathcal{H} \otimes K$ be the tensor product of $\mathcal{H}$ and $K$. We shall consider the $W^*$-representation $\{\pi \otimes I_K, \mathcal{H} \otimes K\}$ of $M$ on $\mathcal{H} \otimes K$, where $\pi \otimes I_K$ is the amplification of $\pi$ (cf. [1]) and $I_K$ is the identity operator on $K$.

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Lemma 2. The commutant \( \{ \pi(M) \otimes I_K \}' = \pi(M)' \otimes B(K) \) in \( \mathcal{S} \otimes K \) has the property P, where \( B(K) \) is the \( W^* \)-algebra of all bounded operators on \( K \).

Proof. \( \pi(M)' \) has the property P (cf. [2]). Let \( a \) be an arbitrary element of the \( W^* \)-algebra \( B(\mathcal{S} \otimes K) \) of all bounded operators on the Hilbert space \( \mathcal{S} \otimes K \), and let \( C_1(a) \) be the \( \sigma \)-weakly closed convex envelope of \( \{ u^*aau \mid u \text{ (unitary)} \in I_b \otimes B(K) \} \), where \( I_b \) is the identity operator on \( \mathcal{S} \). Then \( C_1(a) \cap (I_b \otimes B(K))' = C_1(a) \cap (B(\mathcal{S}) \otimes I_K) \neq (\varnothing) \), because \( I_b \otimes B(K) \) is a type I factor so that there exists an increasing sequence of \( 2^n \times 2^n \) matrix algebras \( \{ M_n \mid n = 1, 2, 3, \ldots \} \) such that the \( \sigma \)-weak closure of \( U_{n-1}M_n = I_b \otimes B(K) \).

Next, take \( a_0 \in C_1(a) \cap (B(\mathcal{S}) \otimes I_K) \), and let \( C_2(a_0) \) be the \( \sigma \)-weakly closed convex envelope of \( \{ u^*a_0au \mid u \text{ (unitary)} \in \pi(M)' \otimes I_K \} \). Then clearly \( C_2(a_0) \cap \{ \pi(M)' \otimes I_K \}' \cap (B(\mathcal{S}) \otimes I_K) \neq (\varnothing) \), because \( \pi(M)' \) has the property P. On the other hand, \( C_2(a_0) \) is contained in the \( \sigma \)-weakly closed convex envelope of \( \{ u^*a_0au \mid u \text{ (unitary)} \in \pi(M)' \otimes B(K) \} \). Hence \( \pi(M)' \otimes B(K) \) has the property. This completes the proof.

The commutant \( \{ \pi(N_1) \otimes I_K \}' \) of \( \pi(N_1) \otimes I_K = \pi(N_1)' \otimes B(K) \); hence it is a II\(_{\infty}\)-factor.

Let \( \xi \) be a trace vector for \( M \) in \( \mathcal{S} \) and \( \eta \) be a nonzero vector in \( K \).

Let \( \{ \pi(N_1) \otimes I_K \xi \otimes \eta \} \) be the closed subspace of \( \mathcal{S} \otimes K \) generated by \( \pi(N_1) \otimes I_K \xi \otimes \eta \) and \( E' \) be the orthogonal projection of \( \mathcal{S} \otimes K \) onto \( \{ \pi(N_1) \otimes I_K \xi \otimes \eta \} \). Then the representation \( x \mapsto (\pi(x) \otimes I_K)E'(x \in N_1) \) of \( N_1 \) is standard; hence \( E' \) is a finite projection in \( \{ \pi(N_1) \otimes I_K \}' \).

Let \( \{ E_n \mid n = 1, 2, \ldots \} \) be a sequence of mutually orthogonal, equivalent projections in \( \{ \pi(N_1) \otimes I_K \}' \) such that \( E_n' \sim E' \) and \( \sum_{n=1}^{\infty} E_n' = I_{b \otimes K} \) where \( I_{b \otimes K} \) is the identity operator on \( b \otimes K \).

Now let \( G \) be the countable discrete free group with two generators, \( L^2(G) \) the Hilbert space of all square integrable functions on \( G \) with respect to the Haar measure, \( L^\infty(G) \) the algebra of all bounded functions on \( G \).

Let \( \{ U, L^2(G) \} \) be the left regular representation of \( G \) on \( L^2(G) \), and \( N \) be the \( W^* \)-algebra generated by \( \{ U(g) \mid g \in G \} \).

For \( f \in L^\infty(G) \), we shall define a bounded operator \( T_f \) on \( L^2(G) \) as follows:

\[
T_fh = foh \quad \text{for } h \in L^2(G).
\]

Let \( A = \{ T_f \mid f \in L^\infty(G) \} \), then \( A \) is a commutative \( W^* \)-algebra on \( L^2(G) \).

\( N \) on \( L^2(G) \) is standard; hence \( N \) on \( L^2(G) \) is spatial isomorphic to \( \{ \pi(N_1) \otimes I_K \}' E' \) on \( E' \mathcal{S} \otimes K \).
Since $E'_n \sim E'$ and $\sum_{n=1}^{\infty} E'_n = I_\mathcal{B} \otimes K$, we can express the $W^*$-algebra $\{\pi(N_1) \otimes I_K\}$ as follows:

$$\left\{\pi(N_1) \otimes I_K\right\} = \mathcal{A} \otimes I_K,$$

where $\mathcal{A}$ is a $W^*$-algebra on a some Hilbert space $\mathcal{H}_1$, and $\mathcal{A}$ on $\mathcal{H}_1$ is spatial isomorphic to $N$ on $L^2(G)$, and $K_1$ is a Hilbert space. We shall identify $\mathcal{A}$ with $N$ by a spatial isomorphism.

Now we shall prove the theorem.

**Proof of Theorem.** Now by Lemma 1, there exists a linear mapping $P$ on $B(\mathcal{H} \otimes K)$ to $\pi(N_1) \otimes I_K$ satisfying the properties (1)-(3).

Let $\tau$ be the trace on $\pi(N_1) \otimes I_K$.

For $f \in L^\infty(G)$, put $f_s(t) = f(s^{-1}t)$ for $s, t \in G$, then $T_s h(t) = f_s(t) h(t) = f(s^{-1}t) h(t) = U(s) T_s U(s^{-1}) h(t)$ for $h \in L^2(G)$. Now we shall define a linear functional $\phi$ on $L^\infty(G)$ as follows: $\phi(f) = \tau(P(T_f \otimes I_{K_1}))$ for $f \in L^\infty(G)$. Then $\phi$ is a positive linear functional such that $\phi(1) = 1$.

Moreover,

$$\varphi(f_s) = \tau(P(U(s) T_f U(s^{-1}) \otimes I_{K_1}))$$

$$= \tau(P(U(s) \otimes I_{K_1}) T_f \otimes I_{K_1} U(s^{-1}) \otimes I_{K_1})$$

$$= \tau(U(s) \otimes I_{K_1} P(T_f \otimes I_{K_1}) U(s^{-1}) \otimes I_{K_1})$$

$$= \tau(P(T_f \otimes I_{K_1}))$$

$$= \phi(f) \quad \text{for } f \in L^\infty(G).$$

Hence, $\phi$ will define a left invariant finitely additive measure on $G$. This is a contradiction (cf. [2]) and completes the proof.

Now, the following problem would be very interesting.

**Problem.** Can we conclude that any $\text{II}_1$-subfactor of the hyperfinite $\text{II}_1$-factor is hyperfinite?

**References**


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