SOME INTEGRAL FORMULAS FOR CLOSED HYPERSURFACES IN EUCLIDEAN SPACE

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Introduction. Let V^n be the orientable hypersurface twice differentiably imbedded in a Euclidean space E^{n+1} of $n+1 \geqq 3$ dimensions, and let $k_1, \cdots, k_n$ be the n principal curvatures at a point P of V^n. The rth mean curvature H_r of V^n at the point P is defined to be the rth elementary symmetric function of $k_1, \cdots, k_n$ divided by the number of terms, that is,

$$\binom{n}{r} H_r = \sum_{i_1 < \cdots < i_r} k_{i_1} \cdots k_{i_r}, \tag{1}$$

and $H_0 = 1$. Throughout this paper all Latin indices take the values $1, \cdots, n$. Greek indices the values $1, \cdots, n+1$, and we shall also follow the convention that repeated indices imply summation. Let p denote the oriented distance from a fixed point O in E^{n+1} to the tangent hyperplane $\pi(P)$ of V^n at the point P, and let $d\Omega$ be the area element of V^n at P. Let $e_1, \cdots, e_n$ be an oriented orthonormal frame in the tangent space of the hypersurface V^n at the point P, and denote by z_i the scalar product of e_i and the position vector of the point P with respect to the fixed point O in the space E^{n+1}. The purpose of this paper is to establish the following

THEOREM. *Let V^n be a closed orientable hypersurface twice differentiably imbedded in a Euclidean space E^{n+1} of $n+1 \geqq 3$ dimensions. Then*

$$n \int_{V^n} p^{m-1}(1 + pH_1) d\Omega - (m-1) \int_{V^n} p^{m-2} \sum_{i,j} a_{ij} z_i z_j d\Omega = 0, \tag{2}$$

$$n \int_{V^n} p^{m-1}(H_{n-1} + pH_n) d\Omega - (m-1) \int_{V^n} p^{m-2} H_n \sum_i z_i^2 d\Omega = 0, \tag{3}$$

where m is any real number, and a_{ij} the coefficients of the second fundamental form of the hypersurface V^n at a general point.

These formulas (2), (3) were obtained by Chern [2] for $n=2$, by Hsiung [3], [4] for $m=1$ in a Euclidean space E^{n+1} as well as a Riemannian space R^{n+1}, and jointly by Hsiung and the author [5] in an affine space A^{n+1}.

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Let p' be the foot of the perpendicular from the fixed point O to the tangent hyperplane $\pi(P)$, d_P the distance PP' and ρ_P the normal curvature of the hypersurface V^n in the direction PP'. Then $\boldsymbol{OP}+\boldsymbol{PP'}=\boldsymbol{OP'}$. By taking scalar products of both sides of this vector equation with e_i we obtain $z_i+e_i\cdot\boldsymbol{PP'}=0$, which enables us to write $-\boldsymbol{PP'}=\sum_i z_ie_i$. Hence we arrive at the following geometric meanings of the terms $\sum_i z_1^2$, $\sum_{i,j} a_{ij}z_iz_j$ in the formulas (3), (2):

$$d_p^2 = \sum_i z_i^2, \qquad d_p^2\rho_P = \sum_{i,j} a_{ij}z_iz_j.$$

1. **Preliminaries.** In a Euclidean space E^{n+1} of dimension $n+1\geqq 3$ let us consider a fixed right-handed rectangular frame $Ye_1\cdots e_{n+1}$, where Y is a point in the space E^{n+1}, and $e_1,\cdots,e_{n+1}$ are an ordered set of mutually orthogonal unit vectors such that its determinant is

$$(4)\qquad \left| e_1,\cdots,e_{n+1}\right| = 1,$$

so that

$$(5)\qquad e_ie_j = \delta_{ij},$$

where δ_{ij} are the Kronecker deltas. We also use Y to denote the position vector of the point P with respect to a fixed point O in the space E^{n+1}. Then we can have

$$(6)\qquad dY = \omega^\alpha e_\alpha,$$

$$(7)\qquad de_\alpha = \overset{\beta}{\omega_\alpha} e_\beta,$$

where d denotes the exterior differentiation, and ω^α, ω_α^β are Pfaffian forms. Since $d^2Y=d(dY)=d(de_\alpha)=0$, exterior differentiation of equations (5), (6), (7) gives

$$(8)\qquad \overset{\beta}{\omega_\alpha} + \overset{\alpha}{\omega_\beta} = 0,$$

and the equations of structure of the group of proper motions in the space E^{n+1}:

$$(9)\qquad d\overset{\alpha}{\omega} = \overset{\beta}{\omega} \wedge \overset{\alpha}{\omega_\beta},$$

$$(10)\qquad d\overset{\beta}{\omega_\alpha} = \overset{\gamma}{\omega_\alpha} \wedge \overset{\beta}{\omega_\gamma},$$

where $\wedge$ denotes the exterior product.

Let V^n be a hypersurface twice differentiably imbedded in the space E^{n+1}. Consider the subfamily of frames $Ye_1\cdots e_{n+1}$ satisfying the conditions (i) $Y\in V^n$, (ii) $e_1,\cdots,e_n$ are vectors tangent to V^n at P. Then we have

$$(11)\qquad \omega^{n+1} = 0,$$

and equations (9), (8) give

$$\omega^{i} \wedge \omega_{i}^{n+1} = 0. \tag{12}$$

By a lemma of E. Cartan [1, p. 11] on exterior algebra, equation (12) implies that for each value of i

$$\omega_{i}^{n+1} = a_{ij}\omega^{j}, \tag{13}$$

with $a_{ij}=a_{ji}$. From equations (6), (7), (8), (13) it follows that at the point P of the hypersurface V^n, the first and second fundamental forms are respectively given by

$$dY\cdot dY = \sum_{i=1}^{n} (\omega^i)^2, \tag{14}$$

$$\Phi = - dY\cdot de_{n+1} = a_{ij}\omega^i\omega^j, \tag{15}$$

and the element of area is given by

$$d\Omega = \omega^1 \wedge \cdots \wedge \omega^n. \tag{16}$$

Thus the n principal curvatures $k_1, \cdots, k_n$ of the hypersurface V^n at the point P are roots of the determinant equation

$$| a_{ij} - k\delta_{ij} | = 0. \tag{17}$$

In other words, $k_1, \cdots, k_n$ are the characteristic roots of the matrix $\|a_{ij}\|$.

A principal minor of the matrix $\|a_{ij}\|$ is a minor whose diagonal is part of the main diagonal of the matrix $\|a_{ij}\|$. From a theorem in linear algebra it is known that the rth elementary symmetric function of the characteristic roots of the matrix $\|a_{ij}\|$ is equal to the sum of all r-rowed principal minors of the matrix $\|a_{ij}\|$. Hence $C_{n,r}H_r$ defined by equation (1) is equal to the sum of all r-rowed principal minors of the matrix $\|a_{ij}\|$. In particular we have

$$nH_1 = \sum_{i=1}^{n} a_{ii}, \tag{18}$$

$$H_n = | a_{ij} |, \tag{19}$$

$$nH_{n-1} = \sum_{i=1}^{n} A^{ii}, \tag{20}$$

where A^{ii} is the cofactor of a_{ii} in the determinant $|a_{ij}|$.

2. **Proof of the Theorem.** Let us now consider the scalar products

$$p = Ye_{n+1}, \qquad z_\alpha = Ye_\alpha. \tag{21}$$

Geometrically, z_α is the oriented distance from the origin O to the hyperplane $Pe_1 \cdots \hat{e}_\alpha \cdots e_{n+1}$, where the circumflex over e_α indicates the vector e_α is to be deleted. In particular z_{n+1} will also be written as p. From equations (21), (7), (8), (13) follows immediately

$$dp = -\sum_i a_{ij}\omega^j z_i. \tag{22}$$

It is convenient to write equations (21) as

$$p = (-1)^n \left| Y, e_1, \cdots, e_n \right|, \tag{23}$$

$$z_\alpha = (-1)^{\alpha-1} \left| Y, e_1, \cdots, \hat{e}_\alpha, \cdots, e_{n+1} \right|. \tag{24}$$

By means of the relation $d^2Y = d^2e_{n+1} = 0$ and the ordinary rule for differentiation of determinants, we have the differential forms

$$\begin{aligned}
&d(p^{m-1} \left| Y, e_{n+1}, dY, \cdots, dY \right|) \qquad \text{(where } dY \text{ occurs } (n-1) \text{ times)}\\
&\quad = (m-1)p^{m-2}dp \wedge \left| Y, e_{n+1}, dY, \cdots, dY \right| \\
&\qquad\qquad \text{(where } dY \text{ occurs } (n-1) \text{ times)}\\
&\qquad - p^{m-1}\left| e_{n+1}, dY, \cdots, dY \right| \quad \text{(where } dY \text{ occurs } n \text{ times)}\\
&\qquad + p^{m-1}\left| Y, de_{n+1}, dY, \cdots, dY \right|, \\
&\qquad\qquad \text{(where } dY \text{ occurs } (n-1) \text{ times)},
\end{aligned} \tag{25}$$

$$\begin{aligned}
&d(p^{m-1} \left| Y, e_{n+1}, de_{n+1}, \cdots, de_{n+1} \right|) \\
&\qquad\qquad \text{(where } de_{n+1} \text{ occurs } (n-1) \text{ times)}\\
&\quad = (m-1)p^{m-2}dp \wedge \left| Y, e_{n+1}, de_{n+1}, \cdots, de_{n+1} \right| \\
&\qquad\qquad \text{(where } de_{n+1} \text{ occurs } (n-1) \text{ times)}\\
&\qquad + p^{m-1}\left| dY, e_{n+1}, de_{n+1}, \cdots, de_{n+1} \right| \\
&\qquad\qquad \text{(where } de_{n+1} \text{ occurs } (n-1) \text{ times)}\\
&\qquad + p^{m-1}\left| Y, de_{n+1}, \cdots, de_{n+1} \right| \\
&\qquad\qquad \text{(where } de_{n+1} \text{ occurs } n \text{ times)}.
\end{aligned} \tag{26}$$

Using the elementary identity $C_{n,r} = n!/[(n-r)!r!]$ and equations (22), (6), (7), (8), (19), (20), (16), (23), (24), from equations (25), (26) we can easily obtain

$$\begin{aligned}
&\frac{(-1)^{n-1}}{(n-1)!} d(p^{m-1} \left| Y, e_{n+1}, dY, \cdots, dY \right|) \\
&\qquad\qquad \text{(where } dY \text{ occurs } (n-1) \text{ times)}\\
&\quad = np^{m-1}(1 + pH_1)d\Omega - (m-1)p^{m-2}\sum_{i,j} a_{ij}z_iz_jd\Omega,
\end{aligned} \tag{27}$$

$$\begin{aligned} &\frac{1}{(n-1)!} d(p^{m-1} \mid Y, e_{n+1}, de_{n+1}, \cdots, de_{n+1} \mid) \\ (28) \qquad &\qquad\qquad \text{(where } de_{n+1} \text{ occurs } (n-1) \text{ times)} \\ &= np^{m-1}(H_{n-1} + pH_n)d\Omega - (m-1)p^{m-2}H_n \sum_i z_i^2 d\Omega. \end{aligned}$$

Application of Stokes' theorem to equations (27), (28) gives immediately the required formulas (2), (3).

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