

# RINGS OF CONTINUOUS FUNCTIONS ON OPEN CONVEX SUBSETS OF $R^n$

LYLE E. PURSELL<sup>1</sup>

**0. Introduction.** If  $h$  is a homeomorphism from a topological space  $X$  onto another space  $Y$ , then one can easily show that the correspondence  $f \rightarrow f \circ h^{-1}$  is an isomorphism from  $C(X)$ , the ring of all real continuous functions on  $X$ , onto  $C(Y)$ , the ring of all real continuous functions on  $Y$ . The converse problem—given an isomorphism  $\phi$  from  $C(X)$  onto  $C(Y)$  to show that there exists a homeomorphism  $h$  from  $X$  onto  $Y$  such that  $\phi f = f \circ h^{-1}$ —is more difficult and is one of the problems which has motivated much of the research on rings of continuous functions. The isomorphism  $\phi$  is usually constructed by using *fixed*, maximal ideals. An ideal  $M$  in  $C(X)$  is fixed if there exists a point  $x_0$  of  $X$  such that  $f(x_0) = 0$  for all  $f$  in  $M$ . If  $X$  is completely regular, then the set  $M(x)$  of all functions in  $C(X)$  which are zero at  $x$  is a fixed, maximal ideal and all fixed, maximal ideals are of this form. If one can show that the property of a maximal ideal being fixed is invariant under ring isomorphisms and if  $X$  and  $Y$  are completely regular, then one can show that the correspondence

$$x \rightarrow M(x) \rightarrow \phi(M(x)) = M(y) \rightarrow y,$$

where  $M(y)$  is the set of all functions in  $C(Y)$  which are zero at a point  $y$  of  $Y$ , is the desired homeomorphism  $h$ . If  $X$  and  $Y$  are compact, then all maximal ideals are fixed, but if  $X$  and  $Y$  are not compact, then there are maximal ideals in  $C(X)$  and  $C(Y)$  which are not fixed. (Nonfixed ideals are called “free” ideals.) The maximal, fixed ideals in  $C(X)$  have been algebraically characterized for certain classes of noncompact spaces,  $X$ , including the class of all separable metric spaces and the class of all normal Hausdorff spaces whose points are  $G$ -delta sets, but these characterizations are complicated and/or difficult to establish. (References: Gelfand and Kolmogoroff [1], Hewitt [4], Pursell [5], and Gillman and Jerison [2].)

In this paper we give a construction of the homeomorphism  $h$  for the case  $X$  and  $Y$  are open convex subsets of  $R^n$  which does not depend on an algebraic characterization of fixed, maximal ideals. This method of construction will also give us the theorem:

---

Presented to the Society, October 5, 1967; received by the editors February 14, 1967.

<sup>1</sup> The author is indebted to the referee for correcting several errors in the original manuscript.

If  $C^\infty(X)$  and  $C^\infty(Y)$  are the rings of all real, infinitely differentiable functions on the open convex subsets  $X$  and  $Y$  of  $R^n$ , respectively and  $\phi$  is an isomorphism from  $C^\infty(X)$  onto  $C^\infty(Y)$ , then there is a diffeomorphism  $h$  from  $X$  onto  $Y$  such that  $\phi f = f \circ h^{-1}$  for all  $f$  in  $C^\infty(X)$ .

Our construction depends upon two properties of infinitely differentiable functions on open convex subsets  $X$  of  $R^n$ :

(0.1) A function in  $C(X)$  can be uniformly approximated on  $X$  by a function in  $C^\infty(X)$ .

(0.2) If  $f$  is in  $C^\infty(X)$ , then to each point  $\bar{a} = (a_1, a_2, \dots, a_n)$  in  $X$ , there corresponds an  $n$ -tuple  $(f_{\bar{a},1}, f_{\bar{a},2}, \dots, f_{\bar{a},n})$  of functions in  $C^\infty(X)$  such that

$$f(\bar{r}) = f(\bar{a}) + \sum_{i=1}^n (r_i - a_i) f_{\bar{a},i}(\bar{r})$$

for all  $\bar{r} = (r_1, r_2, \dots, r_n)$  in  $X$ .

The proof of the first of these properties is given in detail in [6] for  $X = R$  and is outlined there for  $X$  any paracompact differentiable manifold. The second is proved in [3, pp. 9–10]. It is a simple consequence of a theorem concerning the differentiability of the remainder term in Taylor’s formula which was proved earlier for  $R$  and  $R^2$  by H. Whitney [7]. (See also [8].)

**1. Preliminaries.** In the sequel  $X$  and  $Y$  always denote open convex subsets of  $R^n$ . We denote  $n$ -tuples of real numbers or real functions by a bar over a symbol. For example, we denote arbitrary points in  $R^n$  by  $\bar{r} = (r_1, r_2, \dots, r_n)$  and arbitrary elements in  $(C(X))^n$ , that is continuous mappings from  $X$  into  $R^n$ , by  $\bar{f} = (f_1, f_2, \dots, f_n)$  or  $\bar{g} = (g_1, g_2, \dots, g_n)$ . If  $\bar{f}$  is in  $(C(X))^n$  and  $\phi$  is an isomorphism from  $C(X)$  to  $C(Y)$ , then by  $\phi\bar{f}$  we mean the  $n$ -tuple  $(\phi f_1, \phi f_2, \dots, \phi f_n)$  in  $(C(Y))^n$ . The projection mappings or coordinates on  $X$  are denoted by  $x_1, x_2, \dots, x_n$ . That is  $x_i(\bar{r}) = r_i$  for all  $\bar{r}$  in  $X$ . Hence  $\bar{x}$  is the identity mapping on  $X$  and  $\bar{y} = (y_1, y_2, \dots, y_n)$  is the identity mapping on  $Y$ . We denote constant functions by boldface. For example if  $a$  is a real number, then  $\mathbf{a}_X(\bar{r}) = a$  for all  $\bar{r}$  in  $X$  and  $\mathbf{a}_Y(\bar{s}) = a$  for all  $\bar{s}$  in  $Y$ . Observe that  $\mathbf{1}_X$  is the multiplicative identity in  $C(X)$ . The set of all real constants on  $X$  is denoted by  $\mathbf{R}(X)$  and on  $Y$  by  $\mathbf{R}(Y)$ .

**2. Construction of the homeomorphism.** Our main theorem may now be stated:

(2.1) **THEOREM.** *If  $X$  and  $Y$  are open convex subsets of  $R^n$  and  $\phi$  is an isomorphism from  $C(X)$  onto  $C(Y)$ , then  $\phi\bar{x}$  is a homeomorphism from  $Y$  onto  $X$  and  $\phi f = f(\phi\bar{x})$  for all  $f$  in  $C(X)$ .*

Our proof requires several lemmas.

(2.2) LEMMA. *If  $\phi$  is any isomorphism from  $C(X)$  onto  $C(Y)$ , then  $\phi \mathbf{a}_X = \mathbf{a}_Y$  for all  $\mathbf{a}_X$  in  $R(X)$ .*

PROOF. For each point  $\bar{r}$  in  $Y$ , the correspondence  $a \rightarrow (\phi \mathbf{a}_X)(\bar{r})$  is a homomorphism  $\psi_{\bar{r}}$  from  $R$  into  $R$ . Hence  $\psi_{\bar{r}}$  is either the zero homomorphism or the identity mapping on  $R$ . But since  $\mathbf{1}_X$  is the multiplicative identity in  $C(X)$ , then  $\phi \mathbf{1}_X = \mathbf{1}_Y$  and, therefore,  $\psi_{\bar{r}} \mathbf{1} = (\phi \mathbf{1}_X)(\bar{r}) = \mathbf{1}$  for all  $\bar{r}$ . Therefore  $\psi_{\bar{r}}$  must be the identity mapping on  $R$  for all  $\bar{r}$  in  $Y$  (compare [2, p. 23]).

(2.3) LEMMA. *If  $\phi$  is an isomorphism from  $C(X)$  onto  $C(Y)$  and  $\bar{g}$  is in  $(C(X))^n$ , then  $(\phi \bar{g})(Y) = \bar{g}(X)$ .*

PROOF. Let  $\bar{a}$  be any point of  $R^n$  and define

$$f_{\bar{a}} = \sum_{i=1}^n (g_i - \mathbf{a}_{iX})^2.$$

then  $f_{\bar{a}}$  is in  $C(X)$  and has a multiplicative inverse in  $C(X)$  if and only if  $\bar{a}$  is in  $R^n - \bar{g}(X)$ . Hence  $\phi f_{\bar{a}}$  has a multiplicative inverse in  $C(Y)$  if and only if  $\bar{a}$  is in  $R^n - \bar{g}(X)$ . But  $\phi f_{\bar{a}} = \sum_{i=1}^n (\phi g_i - \mathbf{a}_{iY})^2$  has a multiplicative inverse in  $C(Y)$  if and only if  $\bar{a}$  is in  $R^n - (\phi \bar{g})(Y)$ .

(2.4) LEMMA. *If  $\phi$  is an isomorphism from  $C(X)$  onto  $C(Y)$ , then  $\phi(f \circ \bar{g}) = f(\phi \bar{g})$  for all  $f$  in  $C^\infty(X)$  and  $\bar{g}$  in  $(C(X))^n$  such that  $\bar{g}$  maps  $X$  into  $X$ .*

PROOF. By (0.2)

$$f(\bar{r}) = f(\bar{a}) + \sum_i (r_i - a_i) f_{\bar{a}, i}(\bar{r})$$

for all  $\bar{a}, \bar{r}$  in  $X$ . Substituting  $\bar{g}$  for  $\bar{r}$ , we obtain

$$f \circ \bar{g} = f(\bar{\mathbf{a}})_X + \sum_i (g_i - \mathbf{a}_{iX}) f_{\bar{a}, i} \circ \bar{g}.$$

Hence by Lemma (2.2)

$$(\phi(f \circ \bar{g}))(\bar{s}) = f(\bar{a}) + \sum_i [(\phi g_i)(\bar{s}) - a_i] \cdot [(\phi(f_{\bar{a}, i} \circ \bar{g}))(\bar{s})]$$

for all  $\bar{a}$  in  $X$  and  $\bar{s}$  in  $Y$ . Since  $(\phi \bar{g})(\bar{s})$  is in  $X$  by (2.3), then we may set  $\bar{a} = (\phi \bar{g})(\bar{s})$ , obtaining  $(\phi(f \circ \bar{g}))(\bar{s}) = f((\phi \bar{g})(\bar{s}))$  for all  $\bar{s}$  in  $Y$ .

We now use (0.1) to extend (2.4) to all functions  $f$  in  $C(X)$ .

(2.5) LEMMA. *If  $\phi$  is an isomorphism from  $C(X)$  onto  $C(Y)$  and  $\bar{g}$  in  $(C(X))^n$  maps  $X$  into  $X$ , then  $\phi(f \circ \bar{g}) = f(\phi\bar{g})$  for all  $f$  in  $C(X)$ .*

PROOF. By (0.1), for each  $f$  in  $C(X)$  and each  $\epsilon > 0$ , there is a corresponding  $f_\epsilon$  in  $C^\infty(X)$  such that

$$(2.5.1) \quad f_\epsilon(\bar{r}) - \epsilon \leq f(\bar{r}) \leq f_\epsilon(\bar{r}) + \epsilon \quad \text{for all } \bar{r} \text{ in } X.$$

Hence

$$(f_\epsilon \circ \bar{g}) - \epsilon_X \leq f \circ \bar{g} \leq (f_\epsilon \circ \bar{g}) + \epsilon_X.$$

Since a function in  $C(X)$  is nonnegative on  $X$  if and only if it has a square root in  $C(X)$ , the isomorphism  $\phi$  is order-preserving. Therefore by (2.4)

$$(2.5.2) \quad \begin{aligned} f_\epsilon(\phi\bar{g}) - \epsilon_Y &= \phi[(f_\epsilon \circ \bar{g}) - \epsilon_X] \leq \phi(f \circ \bar{g}) \leq \phi[(f_\epsilon \circ \bar{g}) + \epsilon_X] \\ &= f_\epsilon(\phi\bar{g}) + \epsilon_Y \quad \text{for all } \epsilon > 0. \end{aligned}$$

But since  $(\phi\bar{g})(Y) = \bar{g}(X) \subseteq X$ , we may substitute  $\phi\bar{g}$  for  $\bar{r}$  in (2.5.1) to obtain

$$(2.5.3) \quad f_\epsilon(\phi\bar{g}) - \epsilon_Y \leq f(\phi\bar{g}) \leq f_\epsilon(\phi\bar{g}) + \epsilon_Y \quad \text{for all } \epsilon > 0.$$

Comparing (2.5.2) and (2.5.3) we have the desired result.

(2.6) COROLLARY. *If  $\phi$  is an isomorphism from  $C(X)$  onto  $C(Y)$ ,  $\bar{f}$  is a continuous map from  $X$  into  $R^n$ , and  $\bar{g}$  is a continuous map from  $X$  into  $X$ , then  $\phi(\bar{f} \circ \bar{g}) = \bar{f}(\phi\bar{g})$ .*

That  $\phi f = f(\phi\bar{x})$  is now a special case of (2.5). To complete the proof of Theorem 2.1, we show that  $\phi\bar{x}$  is a homeomorphism from  $Y$  onto  $X$ . Clearly  $\phi\bar{x}$  is a continuous map from  $Y$  into  $R^n$ . By (2.3)  $\phi\bar{x}$  maps  $Y$  onto  $X$ . If we interchange  $Y$  and  $X$  in (2.3) and replace  $\phi$  by  $\phi^{-1}$ , then we see that  $\phi^{-1}\bar{y}$  maps  $X$  onto  $Y$ . If we substitute  $\phi^{-1}\bar{y}$  for  $\bar{f}$  and  $\bar{x}$  for  $\bar{g}$  in (2.6), we obtain

$$\phi((\phi^{-1}\bar{y}) \circ \bar{x}) = (\phi^{-1}\bar{y}) \circ (\phi\bar{x}).$$

But since  $\bar{x}$  is the identity map on  $X$ ,

$$\phi((\phi^{-1}\bar{y}) \circ \bar{x}) = \phi(\phi^{-1}\bar{y}) = \bar{y}.$$

Hence  $\phi\bar{x}$  has a continuous inverse  $\phi^{-1}\bar{y}$ . Therefore  $\phi\bar{x}$  is a homeomorphism from  $Y$  onto  $X$  and Theorem 2.1 is proved.

If  $Y = X$  and we replace  $\phi^{-1}$  in the above discussion by an arbitrary automorphism on  $C(X)$ , we obtain  $\phi(\psi\bar{x}) = (\psi\bar{x}) \circ (\phi\bar{x})$ . It follows that the correspondence  $\phi \rightarrow \phi\bar{x}$  is an anti-isomorphism from the group

of all automorphisms on  $C(X)$  onto the group of all homeomorphisms on  $X$ .

In Lemmas (2.2), (2.3), (2.4), and (2.6) we could have replaced the rings  $C(X)$  and  $C(Y)$  by the rings  $C^\infty(X)$  and  $C^\infty(Y)$ , obtaining:

(2.7) LEMMA. *If  $\phi$  is an isomorphism from  $C^\infty(X)$  onto  $C^\infty(Y)$ ,  $\bar{f}$  and  $\bar{g}$  are in  $(C^\infty(X))^n$ , and  $\bar{g}$  maps  $X$  into  $X$ , then*

$$(i) (\phi\bar{f})(Y) = \bar{f}(X),$$

$$(ii) \phi(\bar{f} \circ \bar{g}) = \bar{f}(\phi\bar{g}),$$

*and for any function  $f$  in  $C^\infty(X)$*

$$(iii) \phi f = f(\phi\bar{x}).$$

That  $\phi\bar{x}$  is a diffeomorphism if  $\phi$  is an isomorphism from  $C^\infty(X)$  onto  $C^\infty(Y)$  may be proved in the same way that we proved above that  $\phi\bar{x}$  is a homeomorphism if  $\phi$  is an isomorphism from  $C(X)$  onto  $C(Y)$ . Hence:

(2.8) THEOREM. *If  $X$  and  $Y$  are open convex subsets of  $R^n$  and  $\phi$  is an isomorphism from  $C^\infty(X)$  onto  $C^\infty(Y)$ , then  $\phi\bar{x}$  is a diffeomorphism from  $Y$  onto  $X$  and  $\phi f = f(\phi\bar{x})$  for all  $f$  in  $C(X)$ .*

#### BIBLIOGRAPHY

1. I. Gelfand and A. Kolmogoroff, *On rings of continuous functions on topological spaces*, C. R. (Doklady) Acad. Sci. URSS 22 (1939), 11–15.
2. Leonard Gillman and Meyer Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N. J., 1960.
3. Sigurdur Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.
4. Edwin Hewitt, *Rings of real-valued continuous functions*. I, Trans. Amer. Math. Soc. 64 (1948), 45–99.
5. Lyle E. Pursell, *An algebraic characterization of fixed ideals in certain function rings*, Pacific J. Math. 5 (1955), 963–969.
6. ———, *Uniform approximation of real continuous functions on the real line by infinitely differentiable functions*, Math. Mag. 40 (1967), 263–265.
7. Hassler Whitney, *Differentiability of the remainder term in Taylor's formula*, Duke Math. J. 10 (1943), 153–158.
8. Solution to problem E1789, Amer. Math. Monthly 73 (1966), 779–780.

GRINNELL COLLEGE AND  
UNIVERSITY OF MISSOURI AT ROLLA