ON EXTREMALS OF COMPOSITE VARIATIONAL PROBLEMS

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The author has previously given existence theorems for functionals in the calculus of variations of the form

$$I[y] = \int_{-\infty}^{\infty} F(x, y(x), y'(x), p[y])dx,$$

defined on a class K of absolutely continuous functions y whose derivatives are in $L^1(-\infty, \infty)$ and where p is a map from K into $L^1(-\infty, \infty)$. In this paper we develop, under suitable hypotheses, a necessary condition for extremals of such functionals.

Specifically, let $F(x, y, d, p)$ be a real valued function of class C^1 on R^4. Denote by C_b the normed linear space of bounded continuous functions on $-\infty < x < \infty$ with norm $\|y\| = \sup|y(x)|$. Let A be the set of all bounded functions which are absolutely continuous on every interval and whose derivatives are in $L^1(-\infty, \infty)$. Evidently, A is a linear manifold of C_b.

Now let p be a map from A into $L^1(-\infty, \infty)$, and we assume that

(i) the map p is Fréchet differentiable, and we denote its derivative at y by $p'[y]$;

(ii) given y in A, then there exists a bounded function $K_y(x, t)$ in $L^1(R^2)$ such that

$$p'[y]\phi = \int_{-\infty}^{\infty} K_y(x, t)\phi(t)dt$$

for any ϕ in A.

In view of the integrability of K_y, it is evident that the above expression represents a function in $L^1(-\infty, \infty)$.

Finally let K, the class of admissible functions, consist of all elements of A for which $F(x, y(x), y'(x), p[y])$ is in $L^1(-\infty, \infty)$, and we assume K is nonempty. If boundary conditions are given, we further restrict K to consist of all functions satisfying the previous condition and the boundary conditions.

THEOREM. *Under the assumptions given, suppose that*

(a) *there exist constants* $m > 0$, $k \geqq 1$ *and* B, *such that* $F(x, y, d, p) \geqq m|d|^k - B$ *for every* (x, y, p), *and*

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(b) *there exists a positive function* $M(r)$, $0<r<\infty$, *and a bounded integrable function* $\phi(x)$, $-\infty<x<\infty$, *such that the partial derivatives* F_y, F_d, *and* F_p *satisfy*

$$|F_y|,\ |F_d| \leqq M(r)(1+|d|^k+|p|), \qquad x^2+y^2\leqq r^2,$$
$$|F_p| \leqq \phi(x)(1+|d|^k+|p|), \qquad \text{for any } y.$$

Then if y_0 *is an extremal in* K *for* $I[y]$, *then almost everywhere on any finite interval* $a\leqq x\leqq b$,

$$F_d(x) - \int_a^x\left[\int_{-\infty}^{\infty} K_{y_0}(s,r)F_p(s)ds + F_y(r)\right]dr = C,$$

a constant, where the partial derivatives are evaluated at y_0.

Proof. Given any interval $[a, b]$ and any y in K, then $F(x, y(x), y'(x), p[y])$ is in $L^1(-\infty, \infty)$ hence is in $L^1[a, b]$, and (a) implies y' is in $L^k[a, b]$. Let η be any Lipschitz function having support in $[a, b]$, then, by the properties of the Fréchet differential [**6**, pp. 35–43], we have for any real λ,

$$p[y+\lambda\eta] = p[y] + \lambda p'[y]\eta + o(\|\lambda\eta\|)$$

and

$$p'[y]\eta = \frac{d}{d\lambda}p[y+\lambda\eta]\Big|_{\lambda=0} = \lim_{\lambda\to 0}\frac{p[y+\lambda\eta]-p[y]}{\lambda}.$$

The latter statement follows since existence of $p'[y]$ implies the existence of the Gateaux or weak differential.

If $|\eta(x)|$, $|\eta'(x)|\leqq N$, $a\leqq x\leqq b$, and $|\lambda|\ll 1$, then by (b) the expressions

$$F_y(x, y_0+\lambda\eta, y_0'+\lambda\eta', p[y_0+\lambda\eta])\eta$$

and

$$F_d(x, y_0+\lambda\eta, y_0'+\lambda\eta', p[y_0+\lambda\eta])\eta'$$

are zero outside $[a, b]$ and are dominated on $[a, b]$ by

$$M(R)N[1+|y_0'+\lambda\eta'|^k + |p[y_0+\lambda\eta]|]$$
$$\leqq M(R)N[1+2^{k-1}(|y_0'|^k+N^k) + |p[y_0]| + |p'[y_0]\eta| + o(1)],$$

where $x^2+|y_0(x)+\lambda\eta(x)|^2\leqq R^2$, $a\leqq x\leqq b$.

Furthermore the expression

$$F_p(x, y_0+\lambda\eta, y_0'+\lambda\eta', p[y_0+\lambda\eta])(d/d\lambda)p[y+\lambda\eta]$$

is dominated by the integrable expression

$$|\phi(x)|\,[1 + 2^{k-1}(|y_0'|^k + N^k) + |p[y_0]| \\ + |p'[y_0]\eta| + o(1)](p'[y_0]\eta + o(1)).$$

Therefore $(d/d\lambda)I[y_0+\lambda\eta]|_{\lambda=0}$ exists and equals zero since y_0 is an extremal. Using the representation given for $p'[y_0]\eta$, this gives the expression

$$\int_a^b \left\{\left[F_y(x) + \int_{-\infty}^{\infty} K_{y_0}(s,x)F_p(s)ds\right]\eta(x) + F_d(x)\eta'(x)\right\} dx = 0,$$

where the partials are evaluated at y_0. The interchange of orders of integration is justified since K_{y_0} and η are bounded and F_p is integrable.

If $\zeta(x)$ is any bounded measurable function such that $\int_a^b\zeta(x)dx=0$, then any Lipschitz function η with support in $[a, b]$ is of the form

$$\eta(x) = \int_a^x \zeta(r)dr = -\int_x^b \zeta(r)dr,$$

and therefore the above expression becomes

$$\int_a^b \left\{F_d(x) - \int_a^x \left[\int_{-\infty}^{\infty} K_{y_0}(s,r)F_p(s)ds + F_y(r)\right]dr\right\}\zeta(x)dx = 0.$$

This holds in particular for all C^∞ functions $\zeta(x)$ with support in $[a, b]$ and satisfying the above. We conclude that almost everywhere on $[a, b]$

$$F_d(x) - \int_a^x \left[\int_{-\infty}^{\infty} K_{y_0}(s,r)F_p(s)ds + F_y(r)\right]dr = C,$$

a constant, where the partial derivatives are evaluated at y_0. This completes the proof.

If slightly stronger conditions are imposed on F and p, a smoothness condition of extremals is obtained as follows:

COROLLARY. *Given the hypotheses of the theorem and the initial assumptions, suppose in addition that F is of class C^2, $F_{dd}\neq 0$, and $p[y_0]$ is a continuous function for y_0, an extremal of $I[y]$ in K. Then y_0 is of class C^1.*

PROOF. The necessary condition proved in the theorem may be written

$$F_d(x, y_0(x), y_0'(x), p[y_0]) = \int_a^x Q(r)dr + C \quad \text{a.e.}$$

and the right side is absolutely continuous. By the added hypotheses, it follows that we may solve for $y_0'(x)$ in terms of x, $y_0(x)$, $p[y_0]$, and the right side; hence, $y_0'(x)$ equals a continuous function almost everywhere on $[a, b]$. It follows that $y_0'(x)$ equals a continuous function and hence y_0 is of class C^1.

Finally suppose that p is a map from $L^1(-\infty, \infty)$ into itself and that the functional takes on the form

$$I_1[y] = \int_{-\infty}^{\infty} F(x, y(x), y'(x), p[y'](x))dx.$$

In this case the admissible class K will be the same as before, and we assume the representation

$$p'[y']\phi' = \int_{-\infty}^{\infty} K_y(x, t)\phi'(t)dt,$$

valid for y in K and ϕ in A. As before, we assume $K_y(x, t)$ is bounded and in $L^1(R^2)$. We state the following necessary condition for an extremal: the proof is analogous to that of the theorem and is omitted.

COROLLARY. *Suppose that y_0 in K is an extremal for $I_1[y]$, and the hypotheses of the theorem are satisfied. Then almost anywhere on any finite interval $a \leqq x \leqq b$*

$$F_d(x) + \int_{-\infty}^{\infty} K_{y_0}(r, x)F_p(r)dr - \int_a^x F_y(r)dr = C,$$

a constant, where the partial derivatives are evaluated at y_0.

EXAMPLE. Let $F(x, y, d, p) = (1+d^2)^{1/2} + p^2/(1+x^2)$, and the hypotheses of the theorem are satisfied with $m = k = 1$, $B = 0$, $M(r) = 1$, and $\phi(x) = 1/(1+x^2)$. Let $p[y] = \int_{-\infty}^{\infty} g(x, t)y(t)dt$ where g is a continuous bounded function with compact support in R^2. All the assumptions are satisfied and the necessary condition can be written

$$y_0'(x)[1 + (y_0'(x))^2]^{-1/2} + 2\int_a^x \left\{ \int_{-\infty}^{\infty} \frac{g(s, r)}{1 + s^2} \left[\int_{-\infty}^{\infty} g(s, t)y_0(t)dt \right] ds \right\} dr = C$$

where y_0 is an extremal in K.

For a discussion of similar results related to the ordinary problem of calculus of variations the reader is referred to [**1**, pp. 28–29] and [**5**].

References

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