1. **Introduction.** We continue the study of families of analytic functions characterized by Lipschitz type conditions initiated in *Analyticity. I* [3]. A new property, $R$, is introduced and "smoothing" operators are developed, enabling the abstract theory to give as by-products many results of the concrete theory, including a simplified proof of the analyticity of the elements of mean value families [2], and the solution of Neumann problem for the sphere in potential theory. Employing the smoothing operators another proof is obtained for the existence of derivatives in the finite dimensional case.

A family of functions $F$ on subsets of a linear space $B$ is called an $R$ family if $F$ is invariant under transformations of $B$ of the form $x \rightarrow rx, \ r > 0$.

2. **Notation and definitions.** Let $B$ and $C$ be Banach spaces, and $F$ a family of continuous functions on open subsets of $B$ into $C$. Let $R$ denote the reals and $\omega$ the positive integers. $F$ is called a $T$ family if for $f, g \in F, r \in R, x \in B$, and $S$ an open set in $B$, $F$ contains $rf$, the function $f+g$ defined on $\text{dom } f \cap \text{dom } g$, the restriction $f|S$ of $f$ to $S$, and the translate $f_x$, where $f_x(y) = f(y - x)$ for $y \in \{x + t; \ t \in \text{dom } f\}$.

$F$ is called an $R$ family if for $f \in F, r > 0$, $F$ contains the function $g$ such that $g(x) = f(rx)$ for $x \in \{t/r; \ t \in \text{dom } f\}$.

$F$ is called an $L$ family if $F$ is a $T$ family and if for all $\delta > 0$, there exists $N(\delta) > 0$, such that $f \in F$, $M > 0$, $x \in B$, $U_x(\delta) = \{y \in B; \|y - x\| < \delta\} \subseteq \text{dom } f$, $\|f(y)\| \leq M$ for $y \in U_x(\delta)$, implies

$$\|f(y) - f(x)\| \leq N(\delta) M \|y - x\|.$$  

(1)

If in (1) "$N(\delta)$" is replaced by "$N/\delta$" $F$ is called an $L_N$ family.

$F$ is said to be closed if for all sequences $f_1, f_2, \cdots$ in $F$ with a common domain $S$, which converge uniformly on $S$ to a limit function $f_0$, $f_0 \in F$. For $\delta > 0$, set $U(\delta) = U_0(\delta)$ and $U = U(1)$.

3. **Property $R$.** Property $R$ is possessed by $L_N$ families formed from complex analytic and harmonic functions. There are however important elementary examples which do not possess property $R$; in particular, the family of solutions of the equation "$\Delta f = cf$," $c$ fixed, $c > 0$, or $c < 0$. 

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519
Let $B$ and $C$ be Banach spaces, $N, M > 0$, $F$ an $L_N$ family from $B$ to $C$, and $f \in F$, such that $\text{dom } f = B$, and $\|f(x)\| \leq M$ for all $x \in B$. Then for $x \in B$, $\|f(x) - f(0)\| \leq N M \|x\| \delta^{-1} \to 0$ as $\delta \to \infty$, $\delta > 0$, and hence $\|f(x) - f(0)\| = 0$, and $f$ is constant. Thus $F$ satisfies Liouville's theorem. Since clearly all $RL$ families are $L_N$ families, $N > 0$, all $RL$ families must satisfy Liouville's theorem. Not all $L$ families satisfy Liouville's theorem, however. A trivial example is the $L$ family from $R$ to $R$ generated by the sine or cosine function.

We now proceed to the study of functions contained in an $RL$ family formed by differentiation and integration of elements of the family.

**Theorem 1.** Let $B$ and $C$ be Banach spaces, $n \in \omega$, $F$ a closed $TR$ family from $B$ to $C$, $\rho > 0$, and $f \in F$, such that for $i = 1, \cdots, n + 1$, $f^{(i)}$ exists and is continuous. For $x \in U(\rho)$, $i = 1, \cdots, n$, set $h_0(x) = f(0)$, and $h_i(x) = f^{(i)}(x, \cdots, x)$. Then $h_i \in F$ for $i = 0, 1, \cdots, n$. Moreover for some $\delta > 0$, $F$ contains the function $\theta$, such that $\theta(x) = f'(x)$ for $x \in U(\delta)$.

We observe from [3] that all $RL$ families satisfy the hypothesis of Theorem 1. We will need the case of $TR$ families in the study of mean value families in §5.

**Proof.** There exists $\delta > 0$ such that $U(\delta) \subseteq \text{dom } f$. For $n \in \omega$, $x \in U(n \delta)$, set $g_n(x) = f(x/n)$.

Then for $p > \rho/\delta$, $p \in \omega$, the sequence $g_p, g_{p+1}, \cdots$ converges uniformly on $U(\rho)$ to $h_0$.

There exists $\delta > 0$, $M > 0$, such that $U(3\delta) \subseteq \text{dom } f$ and $\|f''r\| \leq M$ for $x \in U(3\delta)$. Let $x \in U(3\delta)$ and for $t \in U(3\delta)$, set $g(t) = f(t) - f'(t)(t - x)$. Then $g(x) = f(x)$ and for $t \in U(3\delta)$,

$$
\|g\| = \|f' - f'_x\| \leq \|t - x\| \sup\{|f''r|; r \in [t, x]|}
\leq \|t - x\| M
$$

and

$$
\|f(t) - f(x) - f'_x(t - x)\|
= \|g(t) - g(x)\|
\leq \|t - x\| \sup\{|g'_r|; r \in [t, x]|}
\leq \|t - x\| \|t - x\| M = M \|t - x\|^2.
$$

Let $|a| \leq 2$, and for $x \in U(\delta)$, $n \in \omega$, set $g_n(x) = n[f(ax + x/n) - f(ax)]$. Since $F$ is a $TR$ family, $g_n \in F$ for $n \in \omega$. Then for $x \in U(\delta)$, $n \in \omega$, we have $ax + x/n$, $ax \in U(3\delta)$, and
Thus the sequence \( g_1, g_2, \ldots \) of \( F \) converges uniformly on \( U(\delta) \) to a limit function \( w_0 \), where \( w_0(x) = f'_a(x) \) for \( x \in U(\delta) \). Since \( F \) is closed, \( w_0 \in F \). Let \( r > \rho/\delta \). Then for \( x \in U(\rho) \), \( h_1(x) = f'_0(x) = r f'_0(x/r) = r w_0(x/r) \), and \( h_1 \in F \). Set \( \theta = w_1 \).

For \( |a| \leq 2, x \in U(\delta/3), n \in \omega \), replacing \( g_n(x) \) by \( n \left[ f'_a x/n(x) - f'_a(x) \right] \)
\[ = n \left[ w_{a+1/n}(x) - w_a(x) \right], \]
and setting \( w_{a,n}(x) = f'_{a,n}(x, x) \), we obtain, when \( a = 0, h_2 \in F \). Continuing this process the theorem follows.

**Theorem 2.** Let \( B \) and \( C \) be Banach spaces, \( F \) a closed \( RL \) family from \( B \) to \( C \), and \( f \) a uniformly continuous function on \( U \) to \( C \), such that \( f(0) = 0 \), and \( f \mid U \in F \). Then there exists a uniformly continuous function \( g \) on \( U \) to \( C \), such that \( g \mid U \in F \), \( f'_a(x) = f(x) \) for all \( x \in U \), and \( \lim_{r \to 1} \frac{g(x) - g(rx)}{1-r} = f(x) \) for all \( x \in \overline{U} - U \).

**Proof.** For \( x \in \overline{U}, 0 < r \leq 1 \), set \( h_r(x) = f(rx)/r \), and set \( h_0(x) = f'_0(x) \). Let \( \epsilon > 0 \). Then there exists \( 0 < \rho < 1 \), such that for \( x \in U(\rho) \),
\[ ||f(x) - f(0) - f'_0(x)|| \leq \epsilon ||x||/2. \]
Then for \( 0 < r, s \leq \rho, x \in \overline{U} \), we have \( rx, sx \in U(\rho) \), and
\[ \left\| h_s(x) - h_r(x) \right\| = \left\| f(sx)/s - f(rx)/r \right\|
\leq s^{-1} \left\| f(sx) - f'_0(sx) \right\| + r^{-1} \left\| f(rx) - f'_0(rx) \right\|
\leq s^{-1} \left[ \epsilon ||sx||/2 \right] + r^{-1} \left[ \epsilon ||rx||/2 \right]
= \epsilon/2 + \epsilon/2 = \epsilon. \]

Since \( f \) is uniformly continuous, there exists \( 0 < \delta < \rho \), such that for \( \rho \leq r \leq s \leq 1, |s-r| \leq \delta \), and \( x \in \overline{U} \), we have \( ||h_s(x) - h_r(x)|| \leq \epsilon. \)

For \( x \in \overline{U} \), set \( g(x) = \int_0^1 h_r(x) \, dr \), (1). For \( x \in U, n > 1/\delta, n \in \omega, \)
\[ \left\| g(x) - \sum_{0}^{n-1}(1/n) h_{i/n}(x) \right\|
\leq \sum_{0}^{n-1} \left\| \int_{i/n}^{(i+1)/n} h_r(x) \, dr - (1/n) h_{i/n}(x) \right\|
\leq \sum_{0}^{n-1} \left\| \int_{i/n}^{(i+1)/n} [h_r(x) - h_{i/n}(x)] \, dr \right\|
\leq \sum_{0}^{n-1} \epsilon/n = n \left[ \epsilon/n \right] = \epsilon, \]
and thus \( g \mid U \) is the limit on \( U \) of the uniformly convergent sequence \( \left\{ \sum_{0}^{n-1}(1/n) h_{i/n} \mid U; n \in \omega \right\} \) in \( F \). Since \( F \) is closed, \( g \mid U \in F \).

For \( x \in U, x \neq 0, \) and \( n \in \omega, 1/n < 1 - ||x|| \), substituting \( s = r ||x|| \) in
(1), we have 
\[ g(x) = \int_0^{||x||} f(sx/||x||) s^{-1} ds \]
and
\[ g(x + x/n) = \int_0^{||x||/(1+1/n)} f(sx(1 + 1/n)/||x + x/n||) s^{-1} ds \]
\[ = \int_0^{||x||/(1+1/n)} f(sx/||x||) s^{-1} ds; \]
and thus
\[ g'(x) = \lim_{n \to \infty} n [g(x + x/n) - g(x)] \]
\[ = ||x|| \lim_{n \to \infty} ||x/n||^{-1} \int_0^{||x||/(1+1/n)} f(sx/||x||) s^{-1} ds \]
\[ = ||x|| \int_0^{||x||} f(||x/s||) ||x||^{-1} = f(x). \]

**Remark.** Theorem 2 may be used to solve the Neumann problem in potential theory for the sphere once the Dirichlet problem is solved for the sphere. Let \( E \) be a Euclidean space, and let \( \phi \) be a map (continuous function) of \( H = \overline{U} - U \) into \( R \), such that \( \int_H \phi d\mu = 0 \), where \( \mu \) is normalized surface measure on \( H \). Let \( f \) be a map on \( \overline{U} \), such that \( f|U \) is harmonic and \( f|H = \phi \). Then \( f(0) = \int_U f d\mu = \int_H \phi d\mu = 0 \). Since the family of harmonic functions on open subsets of \( E \) into \( R \) is an \( RL \) family, from Theorem 2, there exists a map \( g \) on \( \overline{U} \), such that \( g|U \) is harmonic, and
\[ \lim_{r \to 1} [g(x) - g(rx)]/(1 - r) = f(x) = \phi(x) \text{ for } x \in H. \]

**4. Smoothing operators.** The operators introduced in this section will enable us to approximate elements of a closed \( T \) family \( F \) by elements of \( F \) which are as smooth, i.e., differentiable, as desired. They are similar to averaging operators used in potential theory [6] with the difference that here averages are taken over cubes rather than spheres.

**Definition 1.** Throughout the remainder of this paper \( E \) shall denote a fixed Euclidean space, and \( e_1, \ldots, e_p, p \in \omega \), a fixed orthonormal basis of \( E \). Set \( Q = \{ x \in E; -1/2 \leq [x,e_i] \leq 1/2, i = 1, \ldots, p \} \) and for \( i = 1, \ldots, p \), set \( Q_i = \{ x \in Q; [x,e_i] = 0 \} \).

Let \( f \) be a map of an open set \( S \) in \( E \) into a Banach space \( B \). Then for \( x \in E, a > 0 \), such that \( x + aQ = \{ x + ay; y \in Q \} \subseteq S \), set \( L(f, a)(x) = a^{-p} \int_{x+aq} f(t) dm(t) \), where \( m \) is Lebesgue measure on \( E \).

**Theorem 3.** Let \( B \) be a Banach space, \( F \) a closed \( T \) family from \( E \) to
B, a > 0, and f ∈ F. Set $g = L(f, a)$ and let $H$ be a compact subset of the domain of $g$ with interior $S$. Then $g|S$ lies in $F$ and $g$ is continuously differentiable. If $f$ is continuously differentiable, $g' = [L(f, a)' = L(f', a)]$. Thus $L(g, a)''$ and $L[L(g, a), a]'''$ exist and are continuous.

Moreover, for $0 < s < a$, $f_s = L(f, s)$ converges uniformly on $H$ to $f|H$ as $s \to 0$.

**Proof.** Let $\varepsilon > 0$ and set $M = H + aQ$. Then $M \subseteq \text{dom} f$, and there exists $\delta > 0$, such that $x, y \in M, \|y - x\| < \delta$, implies $\|f(y) - f(x)\| \leq \varepsilon$. Let $0 < r < \delta/p^{1/2}$, and $t_1, \ldots, t_n \in aQ, n \in \omega$, such that $\{t_i + rQ; i = 1, \ldots, n\}$ is a subdivision of $aQ$. Then for $x \in H$, $i = 1, \ldots, n$, $t \in t_i + rQ$, we have $x + t, x + t_i \in x + t_i + rQ \subseteq x + aQ \subseteq H + aQ = M$, $\|f(x + t) - (f(x) + t_i)\| = \|t - t_i\| \leq p^{1/2}r < \delta$, and $\|f(x + t) - f(x + t_i)\| \leq \varepsilon$. Thus for $x \in H$,

$$\left\| \sum_{i=1}^{n} \frac{r}{a} f_{s_i}(x) - g(x) \right\| \leq a^{-p} \left\| \sum_{i=1}^{n} f(x + t_i)r^p - \int_{t_i + rQ} f(x + t)dm(t) \right\| \leq a^{-p} \sum_{i=1}^{n} \int_{t_i + rQ} [f(x + t_i) - f(x + t)]dm(t) \leq a^{-p} \sum_{i=1}^{n} \varepsilon r^p = a^{-p}[\varepsilon a^p] = \varepsilon.$$  

Thus $g|S$ is the uniform limit on $S$ of elements of $F$ of the form $\sum_{i=1}^{n} (r/a) f_{s_i}(x)$. Since $F$ is closed, $g|S \subseteq F$.

Similarly for $s > 0, s > a, \delta,$ and $x \in H$,

$$\|f(x) - f_s(x)\| = s^{-p}\|f_s \int_{sQ} [f(x) - f(x + t)]dm(t)\| \leq s^{-p}[\varepsilon s^p] = \varepsilon,$$

and thus $f_s$ converges uniformly on $H$ to $f$ as $s \to 0, s < a$.

Let $i = 1, \ldots, p$, and set $\rho = a/2$. Then $M_i = H + [-\rho, \rho] e_i \subseteq M_i$. For $x \in M_i$, set $g_i(x) = a^{-p} \int_{aQ_i} f(x + t)dm(t)$. Then for $x, y \in M_i, \|y - x\| < \delta$,

$$\|g_i(y) - g_i(x)\| = a^{-p}\left| \int_{aQ_i} [f(y + t) - f(x + t)]dm(t) \right| \leq a^{-p}[\varepsilon a^{p-1}] = \varepsilon/a.$$

From (1), the function $A: x \to A_x(x \in H)$ is continuous. Then for $x \in S, t \in E$, such that $U_x(||t||) \subseteq S$, setting $s_i = [t, e_i]$ for $i = 1, \ldots, p$, and setting $t_1 = x$, and $t_i = x + \sum_{i=1}^{i-1} s_i e_i$ for $i = 2, \ldots, p + 1$, we have $t_i \in S$ for $i = 1, \ldots, p + 1$, and $\|s_i e_i\| < \delta$ for $i = 1, \ldots, p$ and
\begin{align*}
\|g(x + t) - g(x) - A_x(t)\| &= \left\| \sum_{1}^{p} g(t_{i+1}) - g(t_i) - A_x(t_{i+1} - t_i) \right\| \\
&\leq \sum_{1}^{p} \|g(t_i + s_ie_i) - g(t_i) - A_x(s_ie_i)\| \\
&\leq \sum_{1}^{p} \left\| \int_{-p}^{p+s_i} g_i(t_i + se_i)ds - \int_{-p}^{p} g_i(t_i + se_i)ds - s_i[g_i(x + pe_i) - g_i(x - pe_i)] \right\| \\
&\leq \sum_{1}^{p} \left| s_i \right| (\epsilon/a) + \left| s_i \right| (\epsilon/a) \leq \sum_{1}^{p} 2\|\epsilon/a = 2pa^{a-1}\|t\|,
\end{align*}

and thus \(g'x\) exists and \(g'x = A_x\), and \(g\) is continuously differentiable.

Assume that \(f\) is continuously differentiable and let \(\epsilon > 0\). Then there exists \(\rho > 0\), such that \(x, y \in M, \|y - x\| < \rho\), implies \(\|f(y) - f(x) - f'(y - x)\| \leq \epsilon \|y - x\|\). Then for \(x \in S, t \in U(\rho)\), such that \(x+t \in S\),
\begin{align*}
\|g(x + t) - g(x) - L(f', a)x(t)\| &= \left\| a^{-\rho} \int_{x+aQ} \left[ f(t + s) - f(s) - f'(t) ds \right] dm(t) \right\| \\
&\leq a^{-\rho} \|\epsilon\|t\|a^\rho \| = \epsilon \|t\|,
\end{align*}

and \(g'x = L(f', a)x\).

Remark. Let \(B\) be a Banach space and \(F\) an \(L\) family from \(E\) to \(B\). Then Theorem 3 may be used to give another proof \([3]\) of the existence of derivatives of elements of \(F\). Let \(f \in F\). For \(n \in \omega\), set \(f_n = L(f, 1/2^n)\). Let \(x \in \text{dom} f, \delta > 0\), such that \(H = U_x(\delta) \subseteq \text{dom} f\). Then there exists \(q \in \omega\), such that \(n \geq q, n \in \omega\), implies \(H \subseteq \text{dom} f_q\). From Theorem 3, for \(n \in \omega\), \(f_n\) lies in \(F\) and is differentiable, and the sequence \(f_q, f_{q+1}, \cdots\) converges uniformly on \(H\) to \(f\). From Theorem
3.4 of [3], for $t \in U_\varepsilon(\delta), f'_t$ exists, and the sequence $(f_0)'_t, (f_1)'_t, \cdots$ converges to $f'_t$.

An extension to the infinite dimensional case, when $E$ is replaced by an arbitrary Banach space, is possible but tedious.

A useful provider of examples is the following consequence of Theorem 3.

**Theorem 4.** Let $B$ be a Banach space, and $F$ a closed $T$ family from $E$ to $B$, such that for $x \in E$, $\delta > 0$, $\{ f \mid U_x(\delta); \, U_x(\delta) \subseteq \text{dom} \, f, \, f \in F \}$ is finite dimensional. Then $F$ is an $L$ family.

**Proof.** Let $\delta > 0$, set $H = U(\delta)$, and set $G = \{ L(f, a) \mid H; \, a > 0, \, f \in F, \, H \subseteq \text{dom} \, L(f, a) \}$. Then $G$ is a finite dimensional linear space, and from Theorem 3, the elements of $G$ are continuously differentiable on $H$. Let $g_1, \cdots, g_n, n \in \omega$, be a basis of $G$ and set $N = \sup \{ \| (g_t)_t \| ; \, t \in H \}$. Then $N < \infty$, and for $i = 1, \cdots, n$, $y \in H$,

$$\| g_t(y) - g_t(0) \| \leq \| y \| \sup \{ \| (g_t)_t \| ; \, t \in [0, y] \} \leq \| y \| N.$$

For $f \in G$, there exists a unique sequence of numbers $L_1(f), \cdots, L_n(f)$, such that $f = \sum L_i(f)g_i$. Since $G$ is finite dimensional and $L_1, \cdots, L_n$ are linear $L = \sup \{ \| L_i \| ; \, i = 1, \cdots, n \} < \infty$. Thus for $f \in G, M > 0$, such that $\| f(y) \| \leq M$ for $y \in H$,

$$\| f(y) - f(0) \| = \left\| \sum_{i=1}^n L_i(f) [g_i(y) - g_i(0)] \right\|$$

$$\| f(y) - f(0) \| \leq \sum_{i=1}^n \| L_i(f) \| \| g_i(y) - g_i(0) \|$$

$$\| f(y) - f(0) \| \leq \sum_{i=1}^n [LM] \cdot [N\| y \|] = N(\delta)M\| y \|,$$

where $N(\delta) = nNL$.

Let $f$ be an element of $F$ such that $H \subseteq \text{dom} \, f$. Then from Theorem 3, $f_H$ is the uniform limit on $H$ of elements of $G$, and hence $f$ satisfies (1), and $F$ is an $L$ family.

**Remark.** The argument for Theorem 4 is somewhat simplified if for $E$ and $B$ we take $R$, and for $G$ we take the family of antiderivatives (indefinite integrals) of elements of $F$. In this case [1] all elements of $F$ are linear combinations of expressions of the form $xe^{ax}$, where $p = 0, 1, \cdots$, and $a$ is arbitrary.

**5. Volume mean families.**

**Theorem 5.** Let $\mu$ be a nonnegative Borel measure on $E$, with compact
support $K$, such that $K$ is contained in no proper subspace of $E$, and $\mu(K) = 1$. Let $F$ be the family of all maps $f$ of open sets in $E$ into $R$, such that for $x \in E$, $\delta > 0$, if $x + \delta K \subseteq \text{dom } f$,

$$f(x) = \int_K f(x + \delta t) d\mu(t).$$

Then the elements of $F$ satisfy an elliptic partial differential equation and are analytic.

A proof of this result involving Fourier transforms and the notion of weak solutions of Laplace's equation is given by Friedman and Littman [2]. A considerably simpler argument is given here using the machinery developed in this paper.

**Proof.** Trivially $F$ is a closed TR family. Set $G = \{L \in L(L(f, a), a, a); a > 0, f \in F\}$. From Theorem 3, $G$ is a TR family, $g^{(3)}$ exists and is continuous for $g \in G$, and $f \in F$, $S \subseteq E$, $\overline{S}$ compact, $\overline{S} \subseteq \text{dom } f$, implies $f|S$ lies in the closure $G_0$ of $G$.

Let $H$ be the space of all symmetric bilinear functionals on $E \times E$ into $R$. For $t \in H$, set $P(t) = \int_K \theta(t, x) d\mu(t)$. For $x, y, z \in E$, set $\theta(x, y)(z, w) = [x, z][y, w]$, and set $I'(x, y) = [x, y]' = P(\theta(x, y))$. Since $\mu$ is not supported on any proper subspace of $E$, $[x, x]' > 0$ for all $x \in E, x \neq 0$. Thus $I'$ is an inner product on $E$. Let $E'$ denote the new Euclidean space $\{E, I'\}$ determined by $I'$.

Let $T$ be a unitary transformation (rotation) of $E'$ into itself. Then for $x, y \in E'$, $[Tx, Ty]' = [x, y]'$, and $P(\theta(Tx, Ty)) = [Tx, Ty]' = [x, y]' = P(\theta(x, y))$. Now the collection $\{\theta(x, y); x, y \in E\}$ generates $H$. Thus with respect to $E'$, $P$ is a rotation invariant operator operating on $H^*$, and hence letting $e'_1, \cdots, e'_p$ be an arbitrary orthonormal basis of $E'$, there exists $c \neq 0$ such that $P(\theta(e'_1, e'_1))$ for all $\theta \in H$.

Let $f \in G$, $x \in \text{dom } f$, and $\rho > \sup \{\|t\|; t \in K\}$, and set $h(t) = f''(t, t)$ for $t \in U(\rho)$. From Theorem 1, $h \in F$, and

$$c \sum_1^p \frac{\partial^2 f(x)}{\partial x_i^2} = c \sum_1^p f''(e'_i, e'_j) = P(f''(t, t)) = \int_K f''(t, t) d\mu(t) = \int_K h(t) d\mu(t) = h(0) = f''(0, 0) = 0.$$

Thus the elements of $G$ are harmonic functions with respect to $E'$. From [4], [5], the elements of $G_0$ and hence of $F$ are harmonic functions with respect to $E'$, and hence analytic.
Remarks. Theorems 4 and 5 can also be handled from the standpoint of distribution theory [7]. We consider the family of functions $F$ in question as a family of distributions and take the closure $F_0$ of $F$ in the weak or distribution sense. In Theorem 5, employing property $R$, we observe that $F_0$ contains the functions $h_x(t) = f'_x(t, t)$ which exist at least weakly. We then show that the elements of $F$ satisfy an elliptic partial differential equation at least weakly, and hence strongly.

In Theorem 4 [1], restricting attention to functions with real range, we have that the weak partial derivatives of the elements of $F$ lie in $F_0$. Since $F$ is finite dimensional, $F_0 = F$, and hence the elements of $F$ are strongly differentiable.

The smoothing operators introduced in §4 can be considered as analogues of convolutions of distributions with suitable approximate identity functions.

References


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