THE ASYMPTOTIC BEHAVIOR OF A DECOMPOSITION DUE TO HARZHEIM

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We will write \((x)\) for the fractional part of a real number \(x\) and 
\([x]\) for the greatest integer less than or equal to \(x\) so that 
\((x) = x - [x]\). E. Harzheim has shown [2] the existence for each \(x\) in 
\([0, 1)\) of a sequence of nonnegative integers \(\alpha_2(x), \ldots, \alpha_n(x), \ldots\) with 
\[((x)2^{\alpha_2(x)}3^{\alpha_3(x)} \cdots n^{\alpha_n(x)})\) < \(1/n\).

The \(\alpha_n\) are determined inductively by setting \(\alpha_n(x)\) equal to the 
unique integer \(k\) such that 
\[\sum_{j=1}^{k} \frac{1}{n^j} \leq ((x)2^{\alpha_2(x)} \cdots (n - 1)^{\alpha_{n-1}(x)}) < \sum_{j=1}^{k+1} \frac{1}{n^j}\] 
(taking \(\sum_{j=1}^{0} = 0\)). This is always possible since \(\sum_{j=1}^{\infty} 1/n^j = 1/(n-1)\). 
Conversely, given any sequence \(\alpha_2, \alpha_3, \ldots\) of nonnegative integers 
there is an \(x\) in \([0, 1)\) with \(\alpha_i(x) = \alpha_i\), \(i = 2, 3, \ldots\). We are concerned 
with the probabilistic behavior of the random variables \(\alpha_n\) and \(x_n\), 
\(x_1(x) = x, x_n(x) = n((x)2^{\alpha_2(x)} \cdots n^{\alpha_n(x)})\).

We will write \(\lambda\) for Lebesgue measure on \([0, 1)\). The terms independence, 
random variable, etc., will refer to this probability space.

**Lemma 1.** Each \(x_n\) is uniformly distributed on \([0, 1)\).

**Proof.** This is certainly true for \(x\). If it is true for \(x_n\) then 
\[\lambda\{x \mid x_{n+1}(x) < a}\] 
\[= \lambda \left\{ \bigcup_{j=0}^{\infty} \{x \mid x_{n+1}(x) \leq a, \alpha_{n+1}(x) = j\} \right\} \] 
\[= \sum_{j=0}^{\infty} \lambda \{x \mid x_{n+1}(x) \leq a, \alpha_{n+1}(x) = j\} \] 
\[= \sum_{j=0}^{\infty} \lambda \left\{ x \mid n \sum_{k=1}^{j} \left( \frac{1}{n + 1} \right)^k \leq x_n < n \sum_{k=1}^{j} \left( \frac{1}{n + 1} \right)^k + \frac{an}{(n + 1)^j} \right\} \] 
\[= \sum_{j=0}^{\infty} \frac{an}{(n + 1)^j} = a. \]

Received by the editors March 24, 1967.

1 This research was supported in part by the National Science Foundation Grant GP 6216.
**Lemma 2.**
\[ \lambda \{ x | \alpha_n(x) = j \} = (n - 1)/n^{j+1}, \quad j = 0, 1, \ldots, \]
\[ \lambda \{ x | \alpha_n(x) \geq j \} = 1/n^j, \quad j = 0, 1, \ldots. \]

**Proof.**
\[ \lambda \{ x | \alpha_n(x) = j \} \]
\[ = \lambda \left\{ x \mid (n - 1) \sum_{k=1}^{j} \frac{1}{n^k} \leq x_{n-1} < (n - 1) \sum_{k=1}^{j+1} \frac{1}{n^k} \right\} \]
\[ = \frac{n - 1}{n^{j+1}}. \]

The second formula is an immediate consequence of the first.

**Lemma 3.** \( \alpha_2(x) = \alpha_2, \alpha_3(x) = \alpha_3, \ldots, \alpha_n(x) = \alpha_n \) if and only if
\[ x = \sum_{j=1}^{\infty} \frac{1}{2^j} + 2^{-\alpha_2} \sum_{j=1}^{\infty} \frac{1}{3^j} + \ldots \]
\[ + 2^{1-\alpha_2} \ldots (k - 1)^{1-\alpha_2} \sum_{j=1}^{\alpha_k} \frac{1}{k^j} + \ldots \]
\[ + 2^{-\alpha_2} \ldots (n - 1)^{1-\alpha_2} \sum_{j=1}^{\alpha_n} \frac{1}{n^j} + 2^{-\alpha_2} \ldots (n - 1)^{1-\alpha_n-1} \frac{\epsilon}{n-1} \]
where \( 0 \leq \epsilon < (n-1)/n^{\alpha_n+1} \). In this case \( x_n(x) = n^{\alpha_n} \epsilon/(n-1) \).

**Proof.** For \( n=2 \) it is easily seen that \( x = \sum_{j=1}^{\infty} \frac{1}{2^j} + \epsilon_2 \) where \( 0 \leq \epsilon_2 < 1/2^{\alpha_2+1} \), and that
\[ x_2 = 2((x \cdot 2^{\alpha_2})) = 2^{\alpha_2+1} \epsilon_2. \]

Assuming the formulas of the theorem to hold up to \( n \) and taking \( \alpha_{n+1} \) to be any nonnegative integer we have
\[ ((x^{2^{\alpha_2}} \cdots n^{\alpha_n})) = x_n(x)/n = n^{\alpha_n} \epsilon/(n-1) \]
so that \( \alpha_{n+1}(x) = \alpha_{n+1} \) if and only if
\[ \frac{n - 1}{n^{\alpha_n}} \sum_{j=1}^{\alpha_{n+1}} \left( \frac{1}{n+1} \right)^j \leq \epsilon < \frac{n - 1}{n^{\alpha_n}} \sum_{j=1}^{\alpha_{n+1}+1} \left( \frac{1}{n+1} \right)^j \]
or
\[ \epsilon = \frac{n - 1}{n^{\alpha_n}} \sum_{j=1}^{\alpha_{n+1}} \left( \frac{1}{n+1} \right)^j + \epsilon' \]
where \( \epsilon' \).
Thus
\[ x = \sum_{j=1}^{\alpha_2} \frac{1}{2^j} + \cdots + 2^{-\alpha_2} \cdots n^{-\alpha_n} \sum_{j=1}^{\alpha_n+1} \left( \frac{1}{n+1} \right)^j \]
\[ + 2^{-\alpha_2} \cdots n^{-\alpha_n} \frac{\epsilon_{n+1}}{n} \]

where
\[ 0 \leq \epsilon_{n+1} = \frac{n \epsilon'}{n-1} < \frac{n}{n+1} \left( \frac{1}{n+1} \right)^{\alpha_{n+1}}. \]

Also
\[ x_{n+1}(x) = (n + 1)((x2^{\alpha_2} \cdots (n + 1)^{\alpha_{n+1}})) \]
\[ = (n + 1)(((n + 1)^{\alpha_{n+1}} \epsilon_{n+1}/n)) = (n + 1)^{\alpha_{n+1}}(n + 1)/n\epsilon_{n+1}. \]

**Theorem 1.** The \( \alpha_n \) are independent and \( x_n \) is independent of \( \alpha_1, \ldots, \alpha_n \). The distributions of these random variables are given in Lemmas 1 and 2.

**Proof.** Let \( \Omega \) be the subset of \([0, 1)\) where \( x_n(x) \leq a, \alpha_2(x) = \alpha_2, \ldots, \alpha_n(x) = \alpha_n \) and set
\[ A = \sum_{j=1}^{\alpha_2} \frac{1}{2^j} + \cdots + 2^{-\alpha_2} \cdots (n - 1)^{-\alpha_n-1} \sum_{j=1}^{\alpha_n} \frac{1}{n^j}. \]

Then
\[
\lambda\{\Omega\} = \lambda\left\{\left[ x \mid x = A + 2^{-\alpha_2} \cdots (n-1)^{-\alpha_n-1} \frac{\epsilon}{n-1}, 0 \leq \epsilon \leq a \frac{n-1}{n} n^{-\alpha_n} \right]\right\}
\[= a \frac{1}{n} 2^{-\alpha_2} \cdots n^{-\alpha_n} = a \prod_{j=2}^{n} \left( \frac{j-1}{j^{\alpha_{j+1}}} \right) \]
\[= \lambda\{[x \mid x_n(x) \leq a]\} \prod_{j=2}^{n} \lambda\{[x \mid \alpha_j(x) = \alpha_j]\}. \]

**Theorem 2.** For almost every \( x \),
(1) \( \alpha_n(x) \) takes only the values 0 and 1 from some point on,
(2) the value 1 occurs infinitely often,
(3) for infinitely many \( n \), \( \alpha_k(x) = 0 \) for \( A_n \leq k < A_{n+1} \), \( A_n \) being any increasing sequence with \( \sum_{n=2}^{\infty} A_n/A_{n+1} = \infty \).
Proof.

\[ \sum_{n=2}^{\infty} \lambda \{ \{x \mid \alpha_n(x) \geq 2\} \} = \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty \]

so according to the Borel-Cantelli theorem the event \( \alpha_n(x) \geq 2 \) can occur only finitely often for almost every \( x \). If only finitely many 1's occur then \( \alpha_n(x) = 0 \) from some point on and \( x \) must be rational.

If we set

\[ \Omega_n = \{ x \mid \alpha_k(x) = 0, \, A_n \leq k < A_{n+1} \} \]

then the sets \( \Omega_n \) are independent and

\[ \lambda \{ \Omega_n \} = \prod_{k=A_n}^{A_{n+1}-1} \frac{k - 1}{k} = \frac{A_n - 1}{A_{n+1} - 1} . \]

Thus \( \sum_{n=0}^{\infty} \lambda \{ \Omega_n \} = \infty \), so by the Borel-Cantelli theorem almost every \( x \) is in infinitely many \( \Omega_n \).

If we set \( Q_n(x) = 2^{a_n} \cdots n^{a_n} \) then, for some integer \( P_n(x) \),

\[ x = P_n(x)/Q_n(x) + (1/n)x_n/Q_n(x). \]

Our final theorem concerns the asymptotic distribution of \( x_n \) and \( Q_n \).

**Theorem 3.** For every \( n \)

\[ x = P_n(x)/Q_n(x) + (1/n)x_n(x)/Q_n(x) \]

where

(i) \( P_n(x) \) is an integer,

(ii) \( Q_n(x) \) is an integer whose prime factors are all \( \leq n \),

(iii) \( x_n(x) \) is uniformly distributed on \([0, 1)\) and is independent of \( Q_n(x) \),

(iv) \[ Q_n(x) = \left( \prod_{j=2}^{n} k^{1/(k-1)} \right) \]

\[ \cdot \exp \left( \theta_n(x) \left[ \sum_{j=2}^{n} (\log^2 k) \left( \frac{1}{k - 1} + \frac{1}{(k - 1)^2} \right) \right]^{1/2} \right) \]

where, as \( n \to \infty \),

\[ \lambda \{ \{x \mid \theta_n(x) \leq a\} \} \to \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{a} e^{-t^2/2} dt. \]

**Proof.** Only (iv) needs to be proved. Straightforward calculations show
\[
\int_0^1 \alpha_k(x) \log k \, dx = \frac{\log k}{k - 1},
\]
\[
\int_0^1 \left( \alpha_k(x) - \frac{1}{k - 1} \right)^2 \log^2 k \, dx = (\log^2 k) \left( \frac{1}{k - 1} + \left( \frac{1}{k - 1} \right)^2 \right),
\]
\[
\int_0^1 \left| \alpha_k(x) - \frac{1}{k - 1} \right|^3 \log^3 k \, dx \leq C \frac{\log^3 k}{k - 1}.
\]

Since
\[
\tau_n^3 = \sum_{k=2}^{n} \int_0^1 \left| \alpha_k(x) - \frac{1}{k - 1} \right|^3 \log^3 k \, dx \leq C_1 \log^4 n
\]
and
\[
\sigma_n^2 = \sum_{k=2}^{n} \int_0^1 \left( \alpha_k(x) - \frac{1}{k - 1} \right)^2 \log^2 k \, dx \geq C_2 \log^3 n,
\]
\[
\tau_n/\sigma_n \to 0 \text{ as } n \to \infty
\]
so we can apply Liapounov's theorem with \(\delta = 1\) [1, Theorem 4.4, p. 144] to conclude that
\[
\theta_n = \frac{1}{\sigma_n} \sum_{k=2}^{n} \left( \alpha_k(x) - \frac{1}{k - 1} \right) \log k
\]
has the asymptotic distribution required in (iv). This gives
\[
\log Q_n = \sum_{k=2}^{n} \frac{\log k}{k - 1} + \sigma_n \theta_n
\]
and
\[
Q_n = \left( \prod_{k=2}^{n} k^{1/(k-1)} \right) \exp(\sigma_n \theta_n).
\]

**Bibliography**


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