THE ASYMPTOTIC BEHAVIOR OF A DECOMPOSITION DUE TO HARZHEIM[1]

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We will write $((x))$ for the fractional part of a real number x and $[x]$ for the greatest integer less than or equal to x so that $((x))=x-[x]$. E. Harzheim has shown [2] the existence for each x in $[0, 1)$ of a sequence of nonnegative integers $\alpha_2(x), \cdots, \alpha_n(x), \cdots$ with

$$((x2^{\alpha_2(x)}3^{\alpha_3(x)} \cdots n^{\alpha_n(x)})) < 1/n.$$

The α_n are determined inductively by setting $\alpha_n(x)$ equal to the unique integer k such that

$$\sum_{j=1}^{k} \frac{1}{n^j} \leqq ((x2^{\alpha_2(x)} \cdots (n-1)^{\alpha_{n-1}(x)})) < \sum_{j=1}^{k+1} \frac{1}{n^j}$$

(taking $\sum_{j=1}^{0}=0$). This is always possible since $\sum_{j=1}^{\infty} 1/n^j = 1/(n-1)$. Conversely, given any sequence $\alpha_2, \alpha_3, \cdots$ of nonnegative integers there is an x in $[0, 1)$ with $\alpha_i(x)=\alpha_i$, $i=2, 3, \cdots$. We are concerned with the probabilistic behavior of the random variables α_n and x_n, $x_1(x) = x$, $x_n(x) = n((x2^{\alpha_2(x)} \cdots n^{\alpha_n(x)}))$.

We will write λ for Lebesgue measure on $[0, 1)$. The terms independence, random variable, etc., will refer to this probability space.

LEMMA 1. *Each x_n is uniformly distributed on* $[0, 1)$.

PROOF. This is certainly true for x. If it is true for x_n then

$$\begin{aligned}
&\lambda\{[x \mid x_{n+1}(x) < a]\} \\
&\quad = \lambda\left\{\bigcup_{j=0}^{\infty} [x \mid x_{n+1}(x) \leqq a, \alpha_{n+1}(x) = j]\right\} \\
&\quad = \sum_{j=0}^{\infty} \lambda\{[x \mid x_{n+1}(x) \leqq a, \alpha_{n+1}(x) = j]\} \\
&\quad = \sum_{j=0}^{\infty} \lambda\left\{\left[x \mid n\sum_{k=1}^{j}\left(\frac{1}{n+1}\right)^k \leqq x_n < n\sum_{k=1}^{j}\left(\frac{1}{n+1}\right)^k + \frac{an}{(n+1)^j}\right]\right\} \\
&\quad = \sum_{j=0}^{\infty} \frac{an}{(n+1)^j} = a.
\end{aligned}$$

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LEMMA 2.

$$\lambda\{[x \mid \alpha_n(x) = j]\} = (n-1)/n^{j+1}, \qquad j = 0, 1, \cdots,$$
$$\lambda\{[x \mid \alpha_n(x) \geqq j]\} = 1/n^j, \qquad j = 0, 1, \cdots.$$

PROOF.

$$\begin{aligned}\lambda\{[x \mid \alpha_n(x) = j]\} &= \lambda\left\{\left[x \mid (n-1)\sum_{k=1}^{j}\frac{1}{n^k} \leqq x_{n-1} < (n-1)\sum_{k=1}^{j+1}\frac{1}{n^k}\right]\right\} \\ &= \frac{n-1}{n^{j+1}}.\end{aligned}$$

The second formula is an immediate consequence of the first.

LEMMA 3. *$\alpha_2(x) = \alpha_2$, $\alpha_3(x) = \alpha_3$, $\cdots$, $\alpha_n(x) = \alpha_n$ if and only if*

$$\begin{aligned}x = &\sum_{j=1}^{\alpha_2}\frac{1}{2^j} + 2^{-\alpha_2}\sum_{j=1}^{\alpha_3}\frac{1}{3^j} + \cdots \\ &+ 2^{-\alpha_2}\cdots(k-1)^{-\alpha_{k-1}}\sum_{j=1}^{\alpha_k}\frac{1}{k^j} + \cdots \\ &+ 2^{-\alpha_2}\cdots(n-1)^{-\alpha_{n-1}}\sum_{j=1}^{\alpha_n}\frac{1}{n^j} + 2^{-\alpha_2}\cdots(n-1)^{-\alpha_{n-1}}\frac{\epsilon}{n-1}\end{aligned}$$

where $0 \leqq \epsilon < (n-1)/n^{\alpha_n+1}$. In this case $x_n(x) = n^{\alpha_n}n\epsilon/(n-1)$.

PROOF. For $n=2$ it is easily seen that $x = \sum_{j=1}^{\alpha_2} 1/2^j + \epsilon_2$ where $0 \leqq \epsilon_2 < 1/2^{\alpha_2+1}$, and that

$$x_2 = 2((x \cdot 2^{\alpha_2})) = 2^{\alpha_2+1}\epsilon_2.$$

Assuming the formulas of the theorem to hold up to n and taking α_{n+1} to be any nonnegative integer we have

$$((x2^{\alpha_2}\cdots n^{\alpha_n})) = x_n(x)/n = n^{\alpha_n}\epsilon/(n-1)$$

so that $\alpha_{n+1}(x) = \alpha_{n+1}$ if and only if

$$\frac{n-1}{n^{\alpha_n}}\sum_{j=1}^{\alpha_{n+1}}\left(\frac{1}{n+1}\right)^j \leqq \epsilon < \frac{n-1}{n^{\alpha_n}}\sum_{j=1}^{\alpha_{n+1}+1}\left(\frac{1}{n+1}\right)^j$$

or

$$\epsilon = \frac{n-1}{n^{\alpha_n}}\sum_{j=1}^{\alpha_{n+1}}\left(\frac{1}{n+1}\right)^j + \epsilon'$$

where

$$0 \leqq \epsilon' < (n-1)/n^{\alpha_n}(1/(n+1))^{\alpha_{n+1}+1}.$$

Thus

$$x = \sum_{j=1}^{\alpha_2} \frac{1}{2^j} + \cdots + 2^{-\alpha_2} \cdots n^{-\alpha_n} \sum_{j=1}^{\alpha_{n+1}} \left(\frac{1}{n+1}\right)^j$$
$$+ 2^{-\alpha_2} \cdots n^{-\alpha_n} \frac{\epsilon_{n+1}}{n}$$

where

$$0 \leqq \epsilon_{n+1} = n^{\alpha_n} \frac{n\epsilon'}{n-1} < \frac{n}{n+1}\left(\frac{1}{n+1}\right)^{\alpha_{n+1}}.$$

Also

$$x_{n+1}(x) = (n+1)((x2^{\alpha_2} \cdots (n+1)^{\alpha_{n+1}}))$$
$$= (n+1)(((n+1)^{\alpha_{n+1}}\epsilon_{n+1}/n)) = (n+1)^{\alpha_{n+1}}(n+1)/n\epsilon_{n+1}.$$

THEOREM 1. *The α_n are independent and x_n is independent of $\alpha_1, \cdots, \alpha_n$. The distributions of these random variables are given in Lemmas* 1 *and* 2.

PROOF. Let Ω be the subset of $[0, 1)$ where $x_n(x) \leqq a$, $\alpha_2(x) = \alpha_2, \cdots, \alpha_n(x) = \alpha_n$ and set

$$A = \sum_{j=1}^{\alpha_2} \frac{1}{2^j} + \cdots + 2^{-\alpha_2} \cdots (n-1)^{-\alpha_{n-1}} \sum_{j=1}^{\alpha_n} \frac{1}{n^j}.$$

Then

$$\lambda\{\Omega\} = \lambda\left\{\left[x \mid x = A + 2^{-\alpha_2} \cdots (n-1)^{\alpha_{n-1}} \frac{\epsilon}{n-1},\ 0 \leqq \epsilon \leqq a\frac{n-1}{n} n^{-\alpha_n}\right]\right\}$$
$$= a\frac{1}{n} 2^{-\alpha_2} \cdots n^{-\alpha_n} = a \prod_{j=2}^{n} \left(\frac{j-1}{j^{\alpha_{j+1}}}\right)$$
$$= \lambda\{[x \mid x_n(x) \leqq a]\} \prod_{j=2}^{n} \lambda\{[x \mid \alpha_j(x) = \alpha_j]\}.$$

THEOREM 2. *For almost every x,*

(1) *$\alpha_n(x)$ takes only the values* 0 *and* 1 *from some point on,*

(2) *the value* 1 *occurs infinitely often,*

(3) *for infinitely many n, $\alpha_k(x) = 0$ for $A_n \leqq k < A_{n+1}$, A_n being any increasing sequence with $\sum_{n=2}^{\infty} A_n/A_{n+1} = \infty$.*

PROOF.

$$\sum_{n=2}^{\infty} \lambda\{[x \mid \alpha_n(x) \geqq 2]\} = \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty$$

so according to the Borel-Cantelli theorem the event $\alpha_n(x) \geqq 2$ can occur only finitely often for almost every x. If only finitely many 1's occur then $\alpha_n(x) = 0$ from some point on and x must be rational. If we set

$$\Omega_n = [x \mid \alpha_k(x) = 0,\ A_n \leqq k < A_{n+1}]$$

then the sets Ω_n are independent and

$$\lambda\{\Omega_n\} = \prod_{k=A_n}^{A_{n+1}-1} \frac{k-1}{k} = \frac{A_n - 1}{A_{n+1} - 1}.$$

Thus $\sum_{n=0}^{\infty} \lambda\{\Omega_n\} = \infty$, so by the Borel-Cantelli theorem almost every x is in infinitely many Ω_n.

If we set $Q_n(x) = 2^{\alpha_2} \cdots n^{\alpha_n}$ then, for some integer $P_n(x)$,

$$x = P_n(x)/Q_n(x) + (1/n)x_n/Q_n(x).$$

Our final theorem concerns the asymptotic distribution of x_n and Q_n.

THEOREM 3. *For every n*

$$x = P_n(x)/Q_n(x) + (1/n)x_n(x)/Q_n(x)$$

where

(i) *$P_n(x)$ is an integer,*

(ii) *$Q_n(x)$ is an integer whose prime factors are all $\leqq n$,*

(iii) *$x_n(x)$ is uniformly distributed on $[0, 1)$ and is independent of $Q_n(x)$,*

(iv)
$$Q_n(x) = \left(\prod_{j=2}^{n} k^{1/(k-1)}\right) \cdot \exp\left(\theta_n(x)\left[\sum_{j=2}^{n} (\log^2 k)\left(\frac{1}{k-1} + \frac{1}{(k-1)^2}\right)\right]^{1/2}\right)$$

where, as $n \to \infty$,

$$\lambda\{[x \mid \theta_n(x) \leqq a]\} \to \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{a} e^{-t^2/2} dt.$$

PROOF. Only (iv) needs to be proved. Straightforward calculations show

$$\int_0^1 \alpha_k(x) \log k dx = \frac{\log k}{k-1},$$

$$\int_0^1 \left(\alpha_k(x) - \frac{1}{k-1}\right)^2 \log^2 k dx = (\log^2 k)\left(\frac{1}{k-1} + \left(\frac{1}{k-1}\right)^2\right),$$

$$\int_0^1 \left|\alpha_k(x) - \frac{1}{k-1}\right|^3 \log^3 k dx \leqq C \frac{\log^3 k}{k-1}.$$

Since

$$\tau_n^3 = \sum_{k=2}^n \int_0^1 \left|\alpha_k(x) - \frac{1}{k-1}\right|^3 \log^3 k dx \leqq C_1 \log^4 n$$

and

$$\sigma_n^2 = \sum_{k=2}^n \int_0^1 \left|\left(\alpha_k(x) - \frac{1}{k-1}\right)\right|^2 \log^2 k dx \geqq C_2 \log^3 n,$$

$\tau_n/\sigma_n \to 0$ as $n \to \infty$ so we can apply Liapounov's theorem with $\delta = 1$ [1, Theorem 4.4, p. 144] to conclude that

$$\theta_n = \frac{1}{\sigma_n} \sum_{k=2}^n \left(\alpha_k(x) - \frac{1}{k-1}\right) \log k$$

has the asymptotic distribution required in (iv). This gives

$$\log Q_n = \sum_{k=2}^n \frac{\log k}{k-1} + \sigma_n \theta_n$$

and

$$Q_n = \left(\prod_{k=2}^n k^{1/(k-1)}\right) \exp(\sigma_n \theta_n).$$

Bibliography

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