REGULAR ELEMENTS IN SEMIPRIME RINGS

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In the proof of Goldie's theorem [**1**, Theorem 4.1], one of the crucial steps is to establish that every large right ideal contains a regular element [**1**, Theorem 3.9]. Recently, S. A. Amitsur told one of the authors he had proved, using the weaker conditions of the ACC on left and right annihilators, that every prime ring contains a left regular element a (i.e., the left annihilator a^l of a is zero) and a right regular element b (i.e. the right annihilator b^r of b is zero). In this note, we generalize Amitsur's result as follows:

THEOREM. *In a semiprime ring R with ACC on left and right annihilators, every large right ideal contains a regular element.*

We remark that for semiprime rings, Goldie's conditions on a ring imply ours [**5**, Lemma 2]. However, the converse does not hold, since there exist noncommutative integral domains which do not have a right quotient division ring (see, e.g. [**1**, §6]).

We also remark that although the ring R in our theorem need not have a (classical) right quotient ring, our result shows that R does have a maximal right quotient ring Q (in the sense of Johnson) and for each $q \in Q$ there exist d, $r \in R$, with d regular, such that $qd = r$.

To remove any doubt about terminology, the ACC *on right annihilators* means that the ACC holds in the set of all right ideals of the form $\{z \in R \mid Sz = 0\}$, S any subset of R, and a right ideal is *large* if it intersects every nonzero right ideal nontrivially. The notation $Z_r(R)$ is used for the right singular ideal of R: $Z_r(R) = \{a \in R \mid a^r \text{ large}\}$.

LEMMA. (i) *If a prime ring R has a maximal right annihilator then $Z_r(R) = 0$.* (ii) *If R is a ring having $Z_r(R) = 0$ and the ACC on right annihilators, then $a^r = 0$ whenever aR is a large right ideal of R.*

Part (i) is proved in [**3**, Theorem 1], whereas part (ii) is proved in [**2**, Theorem 3.3].

PROOF OF THE THEOREM. We note that the ACC on right annihilators is equivalent to the DCC on left annihilators. We first prove the theorem in the special case that R is prime.

Let A be a large right ideal of the prime ring R. We shall show that for every $a \in A$ such that either $a^r \neq 0$ or $a^l \neq 0$, there exists $b \in A$

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such that $b^r \subseteq a^r$ and $b^l \subseteq a^l$ with at least one of these inequalities strict. By the DCC on right and left annihilators, it will follow that $a^r = 0$ and $a^l = 0$ for some $a \in A$.

By Zorn's lemma, R contains direct sums of the form $aR \oplus B$, $B \subset A$, and $Ra \oplus C$ (B and C being right and left ideals of R, respectively), where $aR \oplus B$ is a large right ideal of R and $Ra \oplus C$ is a large left ideal.

If $a^r \neq 0$, then by the lemma (and its left-handed version) neither aR nor Ra is large in R; hence B and C are nonzero. The same conclusion holds if $a^l \neq 0$. Since R is prime, $BC \neq 0$ ($BC \subseteq A$) and we have the following nontrivial direct sums of additive groups:

$$(1) \qquad aR \oplus BC \subseteq aR \oplus B, \qquad Ra \oplus BC \subseteq Ra \oplus C.$$

The directness of the sums in (1) shows that for every $y \in BC$

$$(2) \qquad (a + y)^r \subseteq a^r, \qquad (a + y)^l \subseteq a^l.$$

We claim that if $a^r \neq 0$, then the first inequality of (2) is strict for some $y \in BC$. For if $(a+y)^r = a^r$ for every $y \in BC$, then $BCa^r = 0$. Since R is prime and B is a right ideal, necessarily $Ca^r = 0$ and hence $(Ra \oplus C)a^r = 0$. However, this equation contradicts the fact, according to the lemma, that $Z_l(R) = 0$.

Similarly, if $a^l \neq 0$, there exists some $y \in BC$ such that the second inequality of (2) is strict. Thus, the theorem is proved if R is prime.

We now carry out a reduction to the prime case, making use of the theorem [**4**, Theorem 3.13] that a semiprime ring R satisfying the ACC on two-sided annihilator ideals is an irredundant subdirect sum of a finite number of prime rings. That is, there exist prime rings $R_1, \cdots, R_n$ whose direct sum contains R, such that $p_i(R) = R_i$ for each coordinate map $p_i: R \to R_i$, and

$$(3) \qquad R \cap R_i \neq \{0\}, \qquad i = 1 \cdots, n.$$

The ideals $\ker p_i$ are the maximal two-sided annihilator ideals of R [**4**, Theorem 3.2].

Another fact which we will need is that if R has the ACC on left and right annihilators, then each R_i satisfies the same conditions. To prove this, we first observe that each two-sided ideal I of R which is an annihilator right ideal is also an annihilator left ideal. Next, we observe that the inverse image B in R of each annihilator right (left) ideal $\overline{B}$ in $\overline{R} = R/I$ is an annihilator right (left) ideal of R. Thus, if $I = X^r$ and $\overline{B} = \overline{Y}^r$ then $B = (XY)^r$, and similarly for left annihilators. Consequently, each $R_i \cong R/\ker p_i$ has the ACC on left and right annihilators.

Now let A be a large right ideal of R and let $A_i = p_i(A)$. We show that $A \cap A_i$ is a large right ideal of R_i. If B_i is a nonzero right ideal of R_i, then $B_i(R \cap R_i)$ is a nonzero right ideal of both R and R_i since R is prime and $R \cap R_i \neq 0$. Since A is large in R, A meets $B_i(R \cap R_i)$ nontrivially. But every element of this "meet" belongs to A and B_i, hence to R_i, and hence to A_i. Thus $A \cap A_i$ meets B_i as desired.

Now, by the prime case of the theorem, each $A \cap A_i$ contains a regular element a_i of R_i. The element $a_1 + a_2 + \cdots + a_n$ of A is thus regular in R, and the proof of the theorem is complete.

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