REGULAR ELEMENTS IN SEMIPRIME RINGS

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In the proof of Goldie's theorem [1, Theorem 4.1], one of the crucial steps is to establish that every large right ideal contains a regular element [1, Theorem 3.9]. Recently, S. A. Amitsur told one of the authors he had proved, using the weaker conditions of the ACC on left and right annihilators, that every prime ring contains a left regular element \( a \) (i.e., the left annihilator \( a^l \) of \( a \) is zero) and a right regular element \( b \) (i.e. the right annihilator \( b^r \) of \( b \) is zero). In this note, we generalize Amitsur's result as follows:

**Theorem.** In a semiprime ring \( R \) with ACC on left and right annihilators, every large right ideal contains a regular element.

We remark that for semiprime rings, Goldie's conditions on a ring imply ours [5, Lemma 2]. However, the converse does not hold, since there exist noncommutative integral domains which do not have a right quotient division ring (see, e.g. [1, §6]).

We also remark that although the ring \( R \) in our theorem need not have a (classical) right quotient ring, our result shows that \( R \) does have a maximal right quotient ring \( Q \) (in the sense of Johnson) and for each \( q \in Q \) there exist \( d, r \in R \), with \( d \) regular, such that \( qd = r \).

To remove any doubt about terminology, the ACC on right annihilators means that the ACC holds in the set of all right ideals of the form \( \{ z \in R \mid Sz = 0 \} \), \( S \) any subset of \( R \), and a right ideal is large if it intersects every nonzero right ideal nontrivially. The notation \( Z_r(R) \) is used for the right singular ideal of \( R \): \( Z_r(R) = \{ a \in R \mid a^r \text{ large} \} \).

**Lemma.** (i) If a prime ring \( R \) has a maximal right annihilator then \( Z_r(R) = 0 \). (ii) If \( R \) is a ring having \( Z_r(R) = 0 \) and the ACC on right annihilators, then \( a^r = 0 \) whenever \( aR \) is a large right ideal of \( R \).

Part (i) is proved in [3, Theorem 1], whereas part (ii) is proved in [2, Theorem 3.3].

**Proof of the Theorem.** We note that the ACC on right annihilators is equivalent to the DCC on left annihilators. We first prove the theorem in the special case that \( R \) is prime.

Let \( A \) be a large right ideal of the prime ring \( R \). We shall show that for every \( a \in A \) such that either \( a^r \neq 0 \) or \( a^l \neq 0 \), there exists \( b \in A \)
such that $b^r \subseteq a^r$ and $b^l \subseteq a^l$ with at least one of these inequalities strict. By the DCC on right and left annihilators, it will follow that $a^r = 0$ and $a^l = 0$ for some $a \in A$.

By Zorn's lemma, $R$ contains direct sums of the form $aR \oplus B$, $B \subseteq A$, and $Ra \oplus C$ ($B$ and $C$ being right and left ideals of $R$, respectively), where $aR \oplus B$ is a large right ideal of $R$ and $Ra \oplus C$ is a large left ideal.

If $a^r \neq 0$, then by the lemma (and its left-handed version) neither $aR$ nor $Ra$ is large in $R$; hence $B$ and $C$ are nonzero. The same conclusion holds if $a^l \neq 0$. Since $R$ is prime, $BC \neq 0$ ($BC \subseteq A$) and we have the following nontrivial direct sums of additive groups:

(1) $aR \oplus BC \subseteq aR \oplus B$, $Ra \oplus BC \subseteq Ra \oplus C$.

The directness of the sums in (1) shows that for every $y \in BC$

(2) $(a + y)^r \subseteq a^r$, $(a + y)^l \subseteq a^l$.

We claim that if $a^r \neq 0$, then the first inequality of (2) is strict for some $y \in BC$. For if $(a + y)^r = a^r$ for every $y \in BC$, then $BCa^r = 0$. Since $R$ is prime and $B$ is a right ideal, necessarily $Ca^r = 0$ and hence $(Ra \oplus C)a^r = 0$. However, this equation contradicts the fact, according to the lemma, that $Z(R) = 0$.

Similarly, if $a^l \neq 0$, there exists some $y \in BC$ such that the second inequality of (2) is strict. Thus, the theorem is proved if $R$ is prime.

We now carry out a reduction to the prime case, making use of the theorem [4, Theorem 3.13] that a semiprime ring $R$ satisfying the ACC on two-sided annihilator ideals is an irredundant subdirect sum of a finite number of prime rings. That is, there exist prime rings $R_1, \ldots, R_n$ whose direct sum contains $R$, such that $\hat{p}_i(R) = R_i$ for each coordinate map $\hat{p}_i: R \rightarrow R_i$, and

(3) $R \cap R_i \neq \{0\}$, $i = 1, \ldots, n$.

The ideals ker $\hat{p}_i$ are the maximal two-sided annihilator ideals of $R$ [4, Theorem 3.2].

Another fact which we will need is that if $R$ has the ACC on left and right annihilators, then each $R_i$ satisfies the same conditions. To prove this, we first observe that each two-sided ideal $I$ of $R$ which is an annihilator right ideal is also an annihilator left ideal. Next, we observe that the inverse image $B$ in $R$ of each annihilator right (left) ideal $\overline{B}$ in $\overline{R} = R/I$ is an annihilator right (left) ideal of $R$. Thus, if $I = X^r$ and $\overline{B} = \overline{Y}^r$ then $B = (XY)^r$, and similarly for left annihilators. Consequently, each $R_i \cong R/\ker \hat{p}_i$ has the ACC on left and right annihilators.
Now let $A$ be a large right ideal of $R$ and let $A_i = p_i(A)$. We show that $A \cap A_i$ is a large right ideal of $R_i$. If $B_i$ is a nonzero right ideal of $R_i$, then $B_i(R \cap R_i)$ is a nonzero right ideal of both $R$ and $R_i$ since $R$ is prime and $R \cap R_i \neq 0$. Since $A$ is large in $R$, $A$ meets $B_i(R \cap R_i)$ nontrivially. But every element of this "meet" belongs to $A$ and $B_i$, hence to $R_i$, and hence to $A_i$. Thus $A \cap A_i$ meets $B_i$ as desired.

Now, by the prime case of the theorem, each $A \cap A_i$ contains a regular element $a_i$ of $R_i$. The element $a_1 + a_2 + \cdots + a_n$ of $A$ is thus regular in $R$, and the proof of the theorem is complete.

**Bibliography**


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