UNCOUNTABLY MANY MILDLY WILD NON-WILDER ARCS

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In [5] Fox and Harrold defined a Wilder arc as a mildly wild L.P.U. (locally peripherally unknotted) arc and then gave a complete classification of such arcs. In this paper, to show that the L.P.U. condition is essential, uncountably many mutually nonequivalent mildly wild non-L.P.U. arcs are constructed.

Figure 1. $A_0$

I. The basic example $A_0$ (Figure 1). A regular normed projection of our basic example of a mildly wild non-Wilder arc is shown in Figure 1. (Using the methods of [4], one could easily give a precise description.)

(A) $A_0$ is not L.P.U. at $p$. The invariants of [7] will be used to show that the penetration index $P(A_0, p)$ of $A_0$ at $p$ is equal to 4. (For a definition of the penetration index see [1] and [2].)

The fundamental group $\pi_1(U_1 - A_0)$ of the complement of $A_0$ in the 3-cell neighborhood $U_1$ of $p$ is generated by elements $a_n, b_n, c_n, d, e_n, f_n, g_n, h_n$ ($n \geq 0$) indicated in the usual way in Figure 1. A set of defining relations is

$$g_n = e_n a_n e_n^{-1}, \quad a_{n+1} = g_n e_n g_n^{-1}, \quad h_n = e_n c_n e_n^{-1}, \quad f_n = h_n e_n h_n^{-1},$$

$$c_{n+1} = e_n b_n e_n^{-1}, \quad a_{n+1} g_n a_{n+1} = c_{n+1} f_n h_n f_n^{-1} c_{n+1}, \quad b_{n+1} = c_{n+1} f_n e_{n+1}, \quad d = a_0^{-1} b_n e_n,$$

where $n \geq 0$. (The methods of [4] were used to find $\pi_1(U_1 - A_0)$.)

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Simplifying the above presentation, one finds that $\pi_1(U_1-A_0)$ is generated by $a_n$, $c_n$, $e_n$ ($n \geq 0$) with the following defining relations:

$$
\begin{align*}
-1 & -1 -1 \\
\epsilon_n a_n e_n a_n & \epsilon_n a_{n+1} = 1, \\
-1 & -1 -1 \\
\epsilon_n & \epsilon_{n+1} c_{n+1} c_n c_n e_n e_n \epsilon_n = 1, \\
-1 & -1 -1 -1 \\
\epsilon_{n+1} c_{n+1} & \epsilon_n c_n c_n c_n e_n \epsilon_n \epsilon_{n+1} e_{n+1} c_n = 1, \\
-1 & -1 -1 \\
\epsilon_{n+1} & \epsilon_n c_n e_n c_n c_n c_n c_n c_n = 1,
\end{align*}
$$

where $n \geq 0$. Hence, the corresponding $JZ$-module is generated by $a_n$, $c_n$, $e_n$ ($n \geq 0$) with the following corresponding defining relations:

$$
\begin{align*}
(t - t^3) a_n + (1 - t + t^3) e_n - a_{n+1} &= 0 \quad (n \geq 0), \\
-t^2 a_n + (t^2 - t^3 + t^4) c_n + (t^3 - t^4) e_n - (1 - t) a_{n+1} + (1 - t) c_{n+1} &= 0, \\
(t^3 - t^4) c_n + (t^2 - t^3 + t^4) e_n + (t - t^2) c_{n+1} + (1 - t) e_{n+1} - e_{n+2} &= 0,
\end{align*}
$$

where $t$ denotes the generator of the free cyclic group $Z$ and $JZ$ denotes the integral group ring of $Z$. Thus, the 0th local topology $\Lambda_0(A_0, p)$ of $A_0$ at $p$ is $\Lambda_0(A_0, p) = (1 - t)^\infty$.

From V.A.7 of [7] the appearance of the factor $1 - t$ in $\Lambda_0(A_0, p)$ implies that $P(A_0, p) > 2$, i.e. that $A_0$ is not L.P.U. at $p$. Moreover, since $P(A_0, p)$ must be even, it easily follows that $P(A_0, p) = 4$. 
(B) $A_0$ is mildly wild.

That $A_0$ is actually mildly wild can be seen by constructing in the obvious fashion an isotopy taking the left half $A_{0L}$ of $A_0$ onto a straight line segment. A more instructive but unfortunately more complicated method for showing that $A_{0L}$ is tame is to observe that its penetration index $[1], [2]$ at $p$ is equal to 1. This can be seen from Figure 2 where a sequence of 2-spheres converging to $p$ and intersecting $A_{0L}$ at only one point is indicated.

II. Uncountably many examples. Let $\Delta_1(t), \Delta_2(t), \Delta_3(t), \cdots$ be an indexing of all mutually distinct (up to associates) prime polynomials in $t$ over the ring $J$ of integers such that:

(i) $\Delta_i(1) = \pm 1$;

(ii) $\Delta_i(t) = t^{2\Delta(1/t)}$.

By [6], [8] there is a corresponding sequence of distinct knot types $K_1, K_2, K_3, \cdots$ whose Alexander polynomials are respectively $\Delta_1, \Delta_2, \Delta_3, \cdots$.

We may assume that each $K_i$ is a prime knot type. For if $K_i$ is not prime, then by [9], [10] it can be uniquely factored into a product of prime knot types, i.e.

$$K_i = K_{i1} \# K_{i2} \# \cdots \# K_{in},$$

where each $K_{ij}$ is prime. Hence,

$$\Delta_i = \Delta(K_i) = \Delta(K_{i1})\Delta(K_{i2}) \cdots \Delta(K_{in}),$$

where $\Delta(K)$ denotes the Alexander polynomial of $K$. Since $JZ$ is a unique factorization domain and $\Delta_i(t)$ is prime, there must be a $j$ such that $\Delta(K_{ij}) = \Delta_i(t)$. Hence, $K_i$ may be replaced by the prime $K_{ij}$.

Let $S$ be the collection of all sequences of 0 or $\omega$, i.e. $S = \{0, \omega\}^N$, where $N$ denotes the set of natural numbers. For each $s$ in $S$ let $W_s$ be the Wilder arc in which each $K_i$ appears exactly $s(i)$ times ($i = 1, 2, 3, \cdots$) and no other knot types appear. (See [5].) Thus, by V.A.7 of [7] the 0th local topology $\Lambda_0(W_s)$ of $W_s$ is

$$\Lambda_0(W_s) = \prod_{i=1}^{\infty} \Delta_i^{s(i)}.$$ 

For each $s$ in $S$, let $A_s = A_0 \# p W_s$, where "$\# p" denotes the interior composition of arcs defined in V.C.7 of [7]. By V.C.11 of [7],

$$\Lambda_0(A_s) = \Lambda_0(A_0) \cdot \Delta_0(W_s) = (1 - t)^{\omega} \prod_{i=1}^{\infty} \Delta_i^{s(i)}.$$
(A) The arcs $A_s$ are all mutually nonequivalent. With the exponent defined in [7] it is easy to show that all the topologies $A_0(A_s)$ are distinct. Hence, by the Fundamental Invariance Theorem of [7] the arcs $A_s$ are all mutually nonequivalent.

(B) Not one of the arcs $A_s$ is L.P.U. As in I.A. of this paper the appearance of $1-t$ in $A_0(A_s)$ implies that $P(A_s) = 4$.

(C) Each of the arcs $A_s$ is mildly wild. $A_s$ can be shown to be mildly wild in the obvious fashion. One can first apply an isotopy to the left half of $A_s$ to straighten out the $A_0$ factor and then another in turn to straighten out the $W_s$ factor. Since the right half of $A_s$ is the right half of a Wilder arc, it is also tame. Hence, $\{A_s: s \in S\}$ is an uncountable collection of mutually nonequivalent mildly wild non-Wilder arcs.

III. Conjecture. The following conjecture is based on the examples in this paper as well as many others.

CONJECTURE. If $A$ is a mildly wild arc, then

$$A_0(A) = (1-t)^{\infty} \prod_{i=1}^{\infty} \Delta_i(t)^{s(i)},$$

where $\Delta_i(t)$ is a knot polynomial and $s(i) = 0$ or $\infty$.

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REFERENCES


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