

UNCOUNTABLY MANY MILDLY WILD NON-WILDER ARCS¹

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In [5] Fox and Harrold defined a *Wilder arc* as a mildly wild L.P.U. (locally peripherally unknotted) arc and then gave a complete classification of such arcs. In this paper, to show that the L.P.U. condition is essential, uncountably many mutually nonequivalent mildly wild non-L.P.U. arcs are constructed.

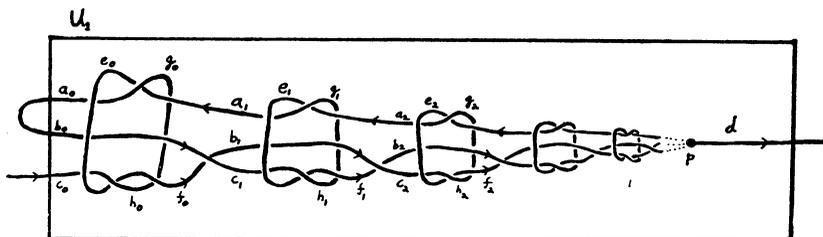


FIGURE 1. A_0

I. **The basic example A_0 (Figure 1).** A regular normed projection of our basic example of a mildly wild non-Wilder arc is shown in Figure 1. (Using the methods of [4], one could easily give a precise description.)

(A) A_0 is not L.P.U. at p . The invariants of [7] will be used to show that the penetration index $P(A_0, p)$ of A_0 at p is equal to 4. (For a definition of the penetration index see [1] and [2].)

The fundamental group $\pi_1(U_1 - A_0)$ of the complement of A_0 in the 3-cell neighborhood U_1 of p is generated by elements $a_n, b_n, c_n, d, e_n, f_n, g_n, h_n$ ($n \geq 0$) indicated in the usual way in Figure 1. A set of defining relations is

$$\begin{aligned} g_n &= e_n a_n e_n^{-1}, & a_{n+1} &= g_n e_n g_n^{-1}, & h_n &= e_n c_n e_n^{-1}, & f_n &= h_n e_n h_n^{-1}, \\ c_{n+1} &= e_n b_n e_n^{-1}, & a_{n+1} g_n a_{n+1}^{-1} &= c_{n+1} f_n h_n f_n^{-1} c_{n+1}^{-1}, & b_{n+1} &= c_{n+1} f_n c_{n+1}^{-1}, \\ d &= a_n^{-1} b_n c_n, \end{aligned}$$

where $n \geq 0$. (The methods of [4] were used to find $\pi_1(U_1 - A_0)$.)

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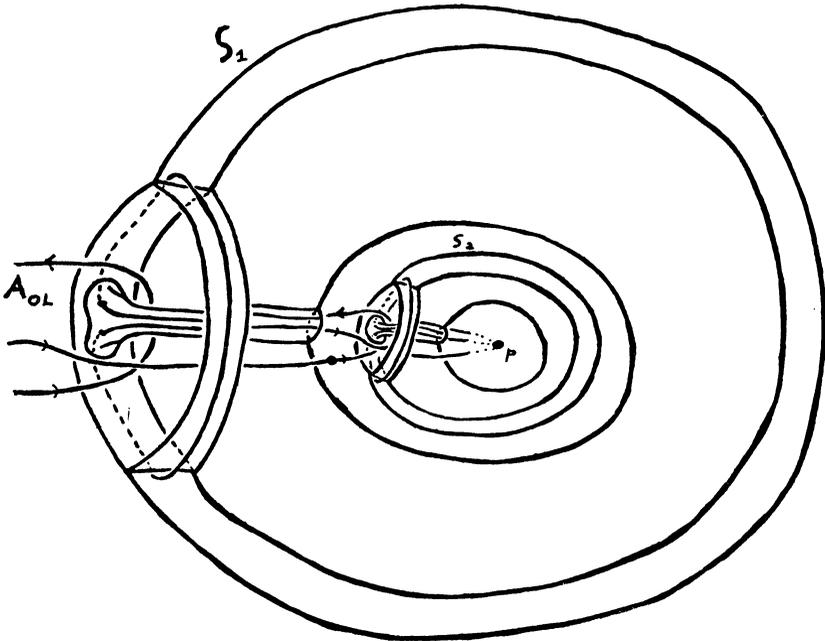


FIGURE 2

Simplifying the above presentation, one finds that $\pi_1(U_1 - A_0)$ is generated by a_n, c_n, e_n ($n \geq 0$) with the following defining relations

$$\begin{aligned}
 e_n a_n e_n a_n e_n a_{n+1} &= 1, \\
 e_n a_{n+1} c_{n+1} e_n c_n e_n c_n e_n c_n e_n c_{n+1} a_{n+1} e_n a_n &= 1, \\
 e_{n+1} c_{n+1} e_n c_n e_n c_n e_n c_{n+1} e_{n+1} c_{n+2} &= 1,
 \end{aligned}$$

where $n \geq 0$. Hence, the corresponding JZ -module is generated by a_n, c_n, e_n ($n \geq 0$) with the following corresponding defining relations:

$$\begin{aligned}
 (t - t^2)a_n + (1 - t + t^2)e_n - a_{n+1} &= 0 \quad (n \geq 0), \\
 -t^2 a_n + (t^2 - t^3 + t^4)c_n + (t^3 - t^4)e_n - (1 - t)a_{n+1} + (1 - t)c_{n+1} &= 0, \\
 (t^3 - t^4)c_n + (t^2 - t^3 + t^4)e_n + (t - t^2)c_{n+1} + (1 - t)e_{n+1} - c_{n+2} &= 0,
 \end{aligned}$$

where t denotes the generator of the free cyclic group Z and JZ denotes the integral group ring of Z . Thus, the 0th local topology $\Lambda_0(A_0, p)$ of A_0 at p is $\Lambda_0(A_0, p) = (1 - t)^\infty$.

From V.A.7 of [7] the appearance of the factor $1 - t$ in $\Lambda_0(A_0, p)$ implies that $P(A_0, p) > 2$, i.e. that A_0 is not L.P.U. at p . Moreover, since $P(A_0, p)$ must be even, it easily follows that $P(A_0, p) = 4$.

(B) A_0 is mildly wild.

That A_0 is actually mildly wild can be seen by constructing in the obvious fashion an isotopy taking the left half A_{0L} of A_0 onto a straight line segment. A more instructive but unfortunately more complicated method for showing that A_{0L} is tame is to observe that its penetration index [1], [2] at p is equal to 1. This can be seen from Figure 2 where a sequence of 2-spheres converging to p and intersecting A_{0L} at only one point is indicated.

II. **Uncountably many examples.** Let $\Delta_1(t), \Delta_2(t), \Delta_3(t), \dots$ be an indexing of all mutually distinct (up to associates) prime polynomials in t over the ring J of integers such that:

- (i) $\Delta_i(1) = \pm 1$;
- (ii) $\Delta_i(t) = t^{2\lambda} \Delta(1/t)$.

By [6], [8] there is a corresponding sequence of distinct knot types K_1, K_2, K_3, \dots whose Alexander polynomials are respectively $\Delta_1, \Delta_2, \Delta_3, \dots$.

We may assume that each K_i is a prime knot type. For if K_i is not prime, then by [9], [10] it can be uniquely factored into a product of prime knot types, i.e.

$$K_i = K_{i1} \# K_{i2} \# \dots \# K_{in},$$

where each K_{ij} is prime. Hence,

$$\Delta_i = \Delta(K_i) = \Delta(K_{i1})\Delta(K_{i2}) \dots \Delta(K_{in}),$$

where $\Delta(K)$ denotes the Alexander polynomial of K . Since JZ is a unique factorization domain and $\Delta_i(t)$ is prime, there must be a j such that $\Delta(K_{ij}) = \Delta_i(t)$. Hence, K_i may be replaced by the prime K_{ij} .

Let S be the collection of all sequences of 0 or ∞ , i.e. $S = \{0, \infty\}^N$, where N denotes the set of natural numbers. For each s in S let W_s be the Wilder arc in which each K_i appears exactly $s(i)$ times ($i = 1, 2, 3, \dots$) and no other knot types appear. (See [5].) Thus, by V.A.7 of [7] the 0th local topology $\Lambda_0(W_s)$ of W_s is

$$\Lambda_0(W_s) = \prod_{i=1}^{\infty} \Delta_i^{s(i)}.$$

For each s in S , let $A_s = A_0 \$_p W_s$, where “ $\$_p$ ” denotes the interior composition of arcs defined in V.C.7 of [7]. By V.C.11 of [7],

$$\Lambda_0(A_s) = \Lambda_0(A_0) \cdot \Lambda_0(W_s) = (1 - t)^\infty \prod_{i=1}^{\infty} \Delta_i^{s(i)}.$$

(A) The arcs A_s are all mutually nonequivalent.

With the exponent defined in [7] it is easy to show that all the topologies $\Lambda_0(A_s)$ are distinct. Hence, by the Fundamental Invariance Theorem of [7] the arcs A_s are all mutually nonequivalent.

(B) Not one of the arcs A_s is L.P.U.

As in I.A. of this paper the appearance of $1-t$ in $\Lambda_0(A_s)$ implies that $P(A_s) = 4$.

(C) Each of the arcs A_s is mildly wild.

A_s can be shown to be mildly wild in the obvious fashion. One can first apply an isotopy to the left half of A_s to straighten out the A_0 factor and then another in turn to straighten out the W_s factor. Since the right half of A_s is the right half of a Wilder arc, it is also tame.

Hence, $\{A_s: s \in S\}$ is an uncountable collection of mutually non-equivalent mildly wild non-Wilder arcs.

III. Conjecture. The following conjecture is based on the examples in this paper as well as many others.

CONJECTURE. If A is a mildly wild arc, then

$$\Lambda_0(A) = (1-t)^\infty \prod_{i=1}^{\infty} \Delta_i(t)^{s(i)},$$

where $\Delta_i(t)$ is a knot polynomial and $s(i) = 0$ or ∞ .

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REFERENCES

1. W. R. Alford and B. J. Ball, *Some almost polyhedral wild arcs*, Duke Math. J., **30** (1963), 33-38.
2. B. J. Ball, "Penetration indices and applications" in *Topology of 3-manifolds and related topics*, Proc. Univ. of Georgia Inst., Prentice-Hall, Englewood Cliffs, N. J., 1962, pp. 37-39.
3. R. H. Fox, *A remarkable simple closed curve*, Ann. of Math. (2) **50** (1949) 264-265.
4. R. H. Fox and E. Artin, *Some wild cells and spheres in three-dimensional space*, Ann. of Math. (2) **49** (1948), 979-990.
5. R. H. Fox and O. G. Harrold, "The Wilder arcs" in *Topology of 3-manifolds and related topics*, Proc. Univ. of Georgia Inst., Prentice-Hall, Englewood Cliffs, N. J., 1962, pp. 184-187.
6. J. Levine, *A characterization of knot polynomials*, Topology **4** (1965), 135-141.
7. S. Lomonaco, *An algebraic theory of local knottedness. I*, Trans. Amer. Math. Soc. **129** (1967), 322-343.
8. H. Seifert, *Über das Geschlecht von Knoten*, Math. Ann. **110** (1934), 571-592.
9. H. Schubert, *Die eindeutige Zerlegbarkeit eines Knotens in Primknoten*, S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **3** (1949), 57-104.
10. ———, *Knoten and Vollringe*, Acta. Math. **90** (1953), 131-286.