ON THE SUBRING GENERATED BY THE
SYMMETRIC ELEMENTS

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J. M. Osborn [3] has characterized those rings with involution J,
with 1, and such that every symmetric element has an inverse under
the additional hypothesis that $S^-$, the subring generated by $S$ (the
set of symmetric elements), is $A$, the ring itself. In this paper we
raise the question as to when $S^- = A$ under a weaker hypothesis and
prove the following theorem. ($A$ henceforth is such that $2A = A$ and
$2x = 0$ implies $x = 0$ for any $x \in A$.)

Theorem 1. Let $A$ be a ring with 1 and suppose that $S$ is a simple
Jordan ring under the Jordan multiplication, $s \circ t = st + ts$ for all $s$ and
t in $S$. Then either $S^- = A$ or $S \subseteq Z$, the center of $A$, or $K$, the set of skew
elements, is an ideal of $A$ with $K^2 \subseteq Z$, $K^3 = \{0\}$, and $S$ is an associative
ring under Jordan multiplication.

We note that Osborn proves that under his hypothesis, either $A$ is
a division ring; the direct sum of two division rings which are anti-
isomorphic, and $J$ interchanges the summands; or $S$ is a field under
Jordan multiplication, $K$ is an ideal of $A$ where $K^2 = \{0\}$. A corollary
to our result is

Corollary 1. Let $A$ be as in Theorem 1. Suppose further that $A$ con-
tains no nilpotent ideals. Then either $A$ is simple or $A$ is a direct sum
of two simple rings which are anti-isomorphic and such that $J$ inter-
changes the direct summands.

In order to prove these results we note quickly

Lemma 1. If $U$ is a proper ideal of $A$ then $U \cap S = \{0\}$.

This follows since $S \cap U$ is a Jordan ideal of $S$ and hence if not zero
is $S$ itself. But $1 \in S$ implies that $U = A$.

We are ready to prove Theorem 1. Herstein [1] has shown that $S^-$
is a Lie ideal of $A$ and Zuev [4] has shown that either $S^-$ is commuta-
tive (in particular $[S, S] = \{0\}$) or $I = \{u \in S^- \mid ua \in S^- \text{ for all } a \in A\}$
is a nonzero two-sided ideal of $S^-$. We next note that $[S, S] \subseteq I$. To

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observe this fact, let \( s, t, u \in S, k \in K \); then most certainly \((st-ts)u \in S^\perp\), while \([s, t]k = s[t, k] + [s, k]t + \{kst - tsk\}\). As the last quantity is in \( S \) we have our desired conclusion. Moreover, \([S, S] \subseteq V = I \cap I'\). Thus, by Lemma 1 we have either \( V = A \) and hence \( A = I \subseteq S^\perp \) (part of our desired conclusion) or \( V \cap S = \{\theta\} \). So far we have shown that either \( S^\perp = A, [S, S] = \{\theta\} \), or \([S, S] \subseteq V\), where \( V \) is an ideal such that \( V \cap S = \{\theta\} \).

We will have almost proved Theorem 1 if we can show that the latter situation \(([S, S] \subseteq V, V \) an ideal with \( V = V' \) and \( V \cap S = \{\theta\}\)) implies either \([S, S] = \{\theta\}\) or the latter possibility in the consequence of the theorem. We now restrict ourselves to this case.

If \( u \in V \) then there exists \( v \in V \) so that \( u = v^I \). Thus, as a consequence of \( V \cap S = \{\theta\} \) we have \( u + v = u + u^I = \theta \), or \( V \subseteq K \). \( V \) is an ideal and so for all \( a \in A, u \in V, ua - a^Iu = \theta \) (in particular, \([S, S] \circ K = \{\theta\}\)). This guarantees that \([S, K] \) is a Jordan ideal of \( S \) since \([S, K] \circ S \subseteq K \circ [S, S] + [S, K]\). Thus, the simplicity of \( S \) implies that either \([S, K] = S \) or \([S, K] = \{\theta\}\).

We now see that the subcase \([S, K] = S \) leads to \([S, S] = \{\theta\}\). To this end, consider \( a, b \in A, v \in V \). Then,

\[
\theta = vab - (ab)^Iv = (va - a^Iv)b + a^I(vb - b^Iv) + (a^Ib^I - b^Ia^I)v
\]

or

\[
[A, A]V = \{\theta\}.
\]

In particular, \([S, K][S, S] = \{\theta\}\) and so \([S, K] = S \) and \( 1 \in S \) yields \([S, S] = \{\theta\}\).

Thus, the subcase \([S, K] = \{\theta\}\) remains. Here we show that either \( S \subseteq Z \) or the latter alternative of the theorem holds. In this case we have \([k^2, s + l] = k \circ [k, s] + [k, k \circ l] = \theta \) for all \( k, l \in K, s \in S \). Thus, \( k^2 \in Z \) for all \( k \in K \). Therefore, \( kl + lk = (k + l)^2 - k^2 - l^2 \in Z \) for all \( k, l \in K \) or \( K \circ K \subseteq Z \). Now, \([K, S] + K \circ K \) is a Jordan ideal of \( S \). However, this is just \( K \circ K \) under our hypothesis. Thus, either \( K \circ K = S \subseteq Z \) (a desired conclusion) or \( K \circ K = \{\theta\} \). The latter situation now concerns us. Now for all \( a \in A, k \in K, ka - a^Ik \in [S, K] + K \circ K = \{\theta\} \). Thus, \( k, l \in K, a \in A \) implies \([kl, a] = (k(a - a^I)l + (ka^I - ak)l = \theta \), or \( K^2 \subseteq Z \). Under our assumptions we also note that \( KA \subseteq K \circ S \subseteq [K, K] \subseteq K \); that is, \( K \) is an ideal of \( A \). \( K \circ K = \{\theta\} \) implies that \( k^2 = \theta \) and \( k \circ l = \theta \) for all \( k, l \in K \). Thus, \( klk = \theta \) and replacing \( k \) by \( k + m \), \( m \) also in \( K \), we have \( km + mlk = \theta \). But \( K \) is an ideal, so \((ml)k + k(ml) = mlk - klm = \theta \). Hence, \( K^2 = \{\theta\} \). Now for all \( s, t, u \in S, (s \circ t) \circ u = s \circ (t \circ u) + [t, [s, u]] \). The latter term being
zero implies that $S$ is an associative ring under the Jordan multiplication.

Therefore, we have shown that either $S^2 = A$, $[S, S] = (\theta)$, or the third consequence in the statement of the theorem holds. What remains is to show that $[S, S] = (\theta)$ implies $S \subseteq Z$. Fix $s \in S$ and consider $[s, K]$. For all $k \in K$, $t \in S$

$$[s, k] \circ t = [s, k \circ t] - k \circ [s, t].$$

By hypothesis the latter term is zero and so $[s, K]$ is a Jordan ideal of $S$. Now, if $[s, K] = (\theta)$ for each $s \in S$ then $S \subseteq Z$ (as $[S, S] = (\theta)$).

We show that the other alternative, $[u, K] = S$ for some $u \in S$ leads to a contradiction. As $[S, S] = (\theta)$ we conclude that $[u, [u, a]] = \theta$ for all $a \in A$. Therefore, replacing $a$ by $ab$ and expanding out (using the fact that $2x = \theta$ implies $x = \theta$) we obtain for all $a, b \in A$

$$(ua - au)(ub - bu) = \theta.$$

Since $1 \in S$ and $[S, S] = (\theta)$, $S^2 = S$ and so $[u, K]^2 = S^2 = S = (\theta)$, a contradiction. Therefore, $S \subseteq Z$ as desired.

The proof of Corollary 1 follows. We first note that the additional hypothesis on $A$ implies that either $S^2 = A$ or $S \subseteq Z$. Now, if $A$ is not simple then we wish to show that the alternative of the conclusion holds. Let $U$ be a proper nonzero ideal of $A$. It readily follows that $T = \{u + u' \mid u \in U\}$ is a Jordan ideal of $S$. Now, if $T \neq (\theta)$ then $S = T$ or $A = U + U'$. We show that indeed the summation is direct and $U$ is a simple ring. By Lemma 1, $U \cap S = (\theta)$. Hence, $V = U \cap U'$ is an ideal with the property $V \cap S = (\theta)$ as well. As before, we conclude that $V \subseteq K$, and $V^2 = (\theta)$. Hence, $V = (\theta)$ or the summation is direct. Now, let $W \neq (\theta)$ be an ideal of $U$. Then, $R = UWU$ is an ideal of $A$, contained in $U$, and nonzero. Else, $(AR)^3 \subseteq UWU = (\theta)$, and this is a contradiction to no nilpotent ideals by Herstein [1]. As before, $A = R + R'$. Therefore, $U \subseteq R \subseteq W$ as $W' \subseteq U'$. Thus, $U$ is simple as desired. Hence, if we show $T = (\theta)$ is impossible we are done. Now, the latter implies that $U \subseteq K$, and analogous to a previous argument $U^3 = (\theta)$. But the additional hypothesis forcing $U = (\theta)$ makes this impossible.

Corollary 2. Let $A$ be without nilpotent ideals. Then either $S^2 = A$ or $A$ is simple and $[A : Z] \leq 4$ or $A = U + U'$, where $U$ is simple and $[U : Z] \leq 4$.

We have shown in Theorem 1 that either $S^2 = A$ or $S \subseteq Z$. Now suppose the latter. Then given any $r \in A$ there exists $s \in S$, $k \in K$ so
that \( r = s + k \). Thus \( r^2 - 2sr = k^2 - s^2 \in \mathbb{Z} \), or each element in \( A \), satisfies a quadratic equation over \( \mathbb{Z} \). The latter, together with Corollary 1, by the work of Kaplansky [2], yields the desired conclusion.

In addition if \( A \) is prime then \( A \) cannot be written as \( U + U^J \) and so \( A \) is simple.

We assume \( 1 \notin A \). Then certain of the results carry over and others do not. The absence of \( 1 \) does not allow us to conclude that if \( U \), an ideal, is such that \( U \supseteq S \) then \( U = A \). However, we do have in this situation that both \( a + a^J \) and \( aa^J \) are in \( U \) for all \( a \in A \). Thus \( a^2 \in U \) for all \( a \in A \), or \( A^2 \subseteq U \) (for example, \( A^2 = A \) implies that \( A = U \)). On the other hand, if \( U \cap S = \{ \theta \} \), then \( V = U \cup U^J \subseteq K \) and \( V^3 = \{ \theta \} \).

Let \( T = \{ u + u^J \mid u \in U \} \) is either \( \{ \theta \} \) or \( S \). If \( T = \{ \theta \} \) then \( U \subseteq K \) and \( U^3 = \{ \theta \} \); while \( T = S \) implies that \( A^3 \subseteq U + U^J \). Now, if we restrict our attention to the ideal \( I \) as defined previously in relation to \( T \) we have either \( S \subseteq I + I^J \subseteq S^J \), which says that \( S^J \) is an ideal or \( T = \{ \theta \} \); that is, \( I \subseteq K \), \( I \cap S = \{ \theta \} \), and \( I^3 = \{ \theta \} \). As in the proof of Theorem 1 this implies that either \( [S, K] = S \) or \( [S, K] = \{ \theta \} \). As before, \( [S, K] = S \) yields \( [S, S] = S[S, S] = \{ \theta \} \) and \( [S, S]K \subseteq [S, S] \circ K + [[S, S], K] \subseteq [S, S] \) (as \( [S, S] \circ K = \{ \theta \} \)). Thus, in this case, \( [S, S] \) is an ideal with \( [S, S]^3 = \{ \theta \} \). On the other hand, \( [S, K] = \{ \theta \} \) yields the same argument as before. We summarize these remarks as

**Theorem 2.** Let \( A \) be a ring with involution and suppose that \( S \) is simple Jordan. Then either \( S \) is an ideal (of \( A \)) containing \( A^3 \), or \( [S, S] \) is an ideal with \( [S, S]^3 = \{ \theta \} \), or \( K \) is an ideal, \( K^2 \subseteq Z \), \( K^3 = \{ \theta \} \) and \( S \) is an associative ring under Jordan multiplication.

Now assume that there are no nilpotent ideals in \( A \); then we have either \( S \) is an ideal or \( [S, S] = \{ \theta \} \). The latter yields, as in a previous argument, that for all \( a, b \in A \), \( u \in S \),

\[
(ua - au)(ub - bu) = \{ \theta \},
\]

and replacing \( b \) by \( ba \) and expanding we have \( \{ (ua - au)A \}^2 = \{ \theta \} \). But by the hypothesis and Herstein [1] we conclude that \( S \subseteq Z \). Thus, either \( S \) is an ideal or \( S \subseteq Z \) and every \( a \in A \) satisfies a quadratic equation over \( Z \). Now if \( U \) is any proper nonzero ideal, then \( S = \{ u + u^J \mid u \in U \} \) and so \( a + a^J = u + u^J \) for each \( a \in A \) and suitable \( u \in U \). Therefore \( a - u \in K \) and so \( A = U + K \). If \( U \cap S = \{ \theta \} \) then, under these hypotheses, \( U \cap K = \{ \theta \} \) and so the group sum is direct.

Finally, if \( A \) is prime and \( U \) is a nonzero ideal then \( U \) contains \( S \) (and hence the ideal \( S^J \), unless \( [S, S] = \{ \theta \} \)), as the other alternative \( U \cap S = \{ \theta \} \) implies \( U \cap U^J = \{ \theta \} \) or \( UU^J = \{ \theta \} \) which, by the hypothesis, is impossible.
References


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