NOTE ON A THEOREM OF BEURLING

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1. **Introduction.** In [3] Somadasa proved the following theorem:

THEOREM [3, p. 297]. *Let μ be a fixed number greater than or equal to 1. Then, corresponding to each point $e^{i\theta}$ on the unit circle, C, in the complex plane, we can construct a class of Blaschke products with the property that each member of this class has T_μ-limit zero at $e^{i\theta}$. Further, there exists a nonempty subclass of this class with the property that each member of this subclass has T_μ-limit zero at $e^{i\theta}$ for all values of μ.*

The purpose of this note is to prove a (slightly stronger form of a) theorem (Theorem 1, below) of Beurling [4]. (No detailed proof of Beurling's Theorem exists in the literature.) This theorem in turn generalizes the result of Somadasa in two directions. First, the class of Blaschke products is replaced by a much larger class of bounded analytic functions. Second, the restriction to T_μ-limits is replaced by essentially arbitrary approaches.

2. **Beurling's Theorem.** Let D denote the open unit disc in the complex plane, C, the unit circle, H^∞, the collection of all bounded analytic functions on D. The pseudo-hyperbolic metric, χ, on D is defined by $\chi(z, w) = |z-w|/|1-\bar{z}w|$. We recall two classical theorems from the theory of complex variables.

PICK'S THEOREM [1, p. 48]. *Let $f \in H^\infty$, and suppose f maps into D. Then for any two points $z, w \in D$ one has*

$$\chi(f(z), f(w)) \leqq \chi(z, w).$$

LINDELÖF'S THEOREM [1, p. 76]. *Let G be a region bounded by a simple closed curve Γ, let $p \in \Gamma$. Let f be continuous on $(G \cup \Gamma) - \{p\}$, bounded and analytic on G. If $f(z)$ approaches the value a at p as z approaches p from either direction on Γ, then $f(z)$ approaches a as z tends to p through $G \cup \Gamma - \{p\}$.*

Recall that a sequence $\{z_n\}$ in D is a Blaschke sequence if and only if $\sum 1 - |z_n|$ converges. With this terminology and the above two theorems we prove

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THEOREM 1. *Let K be a compact subset of $D\cup\Gamma$ such that $K\cap\Gamma = \{e^{i\theta}\}$. Then there exists a Blaschke sequence $\{z_n\}$ in D, $z_n\to e^{i\theta}$ with the property that whenever $f\in H^\infty$ and $f(z_n)\to 0$, then*

$$\lim_{z\to e^{i\theta};z\in K} f(z) = 0.$$

PROOF. We may assume $e^{i\theta}=1$. Let K' be the convex hull of $K\cup\overline{K}$, where $\overline{K}$ is the set of conjugates of the points of K. Let γ' be the boundary of K'. By Lindelöf's Theorem to prove the present theorem it is enough to construct a Blaschke sequence $\{z_n\}$ on γ' with the property that if $f\in H^\infty$ and $f(z_n)\to 0$, then $f(z)\to 0$ as $z\to 1$ on γ'. Now, the union of two Blaschke sequences tending to 1 is again a Blaschke sequence tending to 1 and γ' is symmetric about the real axis. So it is sufficient to find a sequence $\{z_n\}$ on the part, γ, of γ' terminating at 1 which lies above the real axis such that $f(z_n)\to 0$ implies $f(z)\to 0$ as $z\to 1$ along γ. We will use these additional properties of γ: γ is convex —hence, rectifiable; as z proceeds along γ to 1, $|z|$ and $\mathrm{Re}(z)$ increase monotonely to 1. Denote the Euclidean arclength measured along γ between $z, w\in\gamma$ by $\gamma(z, w)$.

With these preliminaries we proceed with the construction of $\{z_n\}$. Choose a point w_1 on γ. Let w_2 be that point on γ satisfying $|w_2|>|w_1|$ and $\gamma(w_1, w_2)=1-|w_2|$. There is such a point since the length of γ exceeds $1-|w_1|$. This same procedure continued indefinitely from w_2 by induction yields a sequence $\{w_n\}$ on γ such that $\gamma(w_n, w_{n+1})=1-|w_{n+1}|$ and $|w_n|\to 1$. The latter follows since $\sum_{n=1}^\infty(1-|w_{n+1}|)=\gamma(w_1, 1)<\infty$, and this also proves that $\{w_n\}$ is a Blaschke sequence. It is easy to find a sequence $\{N_n\}$ of integers with $N_n\to\infty$ while $\sum_{n=1}^\infty N_n(1-|w_{n+1}|)$ is still convergent. Construct a new Blaschke sequence $\{z_k\}$ on γ consisting for each n of the points $w_n = w_{n,0},\ w_{n,1},\ \cdots,\ w_{n,N_n} = w_{n+1}$, where $\gamma(w_{n,j}, w_{n,j+1}) = N_n^{-1}\gamma(w_n, w_{n+1})$, $j=0, \cdots, N_n-1$. Let z be any point on γ between $w_{n,j}$ and $w_{n,j+1}$ inclusive. Then

$$\begin{aligned} N_n|z-w_{n,j+1}| &\leqq N_n\gamma(z, w_{n,j+1}) \leqq N_n\gamma(w_{n,j}, w_{n,j+1}) \\ &= \gamma(w_n, w_{n+1}) = 1-|w_{n+1}| \leqq 1-|w_{n,j+1}|. \end{aligned}$$

Thus,

$$\chi(z, w_{n,j+1}) \leqq \frac{|z-w_{n,j+1}|}{1-|w_{n,j+1}|} \leqq \frac{1}{N_n}.$$

Now, suppose $f\in H^\infty$, $f(z_k)\to 0$, and, without loss of generality, that $|f|$ is bounded by 1. Then, by Pick's Theorem and the last inequality

$$\chi(f(z), f(w_{n,j+1})) \leqq 1/N_n \to 0.$$

Since $f(w_{n,j+1}) \to 0$, $f(z) \to 0$ as $z \to 1$, $z \in \gamma$. This completes the proof.

In particular, the Blaschke product with zeros $\{z_n\}$ tends to zero as $z \to e^{i\theta}$ through K. By definition a function $f \in H^\infty$ has T_μ-limit zero at 1, $\mu \geqq 1$, in case $f(z) \to 0$ as $z \to 1$ through the sets

$$K(m, \mu) = \{z: 1 - |z| \geqq m(\arg z)^\mu, 0 < |z| < 1\}$$

for each $m > 0$. Thus, it is clear how Somadasa's Theorem may be obtained from Theorem 1.

Theorem 1 also points out that the approaches to 1 through the sets $K(m, \mu)$, $m > 0$, $\mu \geqq 1$, fall rather short of exhausting the different possible tangential approaches to 1. For let γ be a convex curve in $D \cup \Gamma$, $\gamma \cap \Gamma = 1$, which is symmetric about the real axis. Then, from the proof of Theorem 1, there is a Blaschke product, $B(z)$, whose zeros are on γ and which tends to zero as $z \to 1$ inside the curve γ. Furthermore, as is well known (e.g., see [2, p. 35]) $B(z)$ tends to each number of modulus not exceeding 1 on some sequence $\{z_n\}$ tending to 1. Each of the sequences must approach 1 more tangentially than the curve γ. This situation persists independent of how large an order of contact at 1 is chosen for γ.

References

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