NOTE ON A THEOREM OF BEURLING

MAX L. WEISS

1. Introduction. In [3] Somadasa proved the following theorem:

Theorem [3, p. 297]. Let \( \mu \) be a fixed number greater than or equal to 1. Then, corresponding to each point \( e^{i\theta} \) on the unit circle, \( C \), in the complex plane, we can construct a class of Blaschke products with the property that each member of this class has \( T_\mu \)-limit zero at \( e^{i\theta} \). Further, there exists a nonempty subclass of this class with the property that each member of this subclass has \( T_\mu \)-limit zero at \( e^{i\theta} \) for all values of \( \mu \).

The purpose of this note is to prove a (slightly stronger form of a) theorem (Theorem 1, below) of Beurling [4]. (No detailed proof of Beurling's Theorem exists in the literature.) This theorem in turn generalizes the result of Somadasa in two directions. First, the class of Blaschke products is replaced by a much larger class of bounded analytic functions. Second, the restriction to \( T_\mu \)-limits is replaced by essentially arbitrary approaches.

2. Beurling's Theorem. Let \( D \) denote the open unit disc in the complex plane, \( C \), the unit circle, \( H^\infty \), the collection of all bounded analytic functions on \( D \). The pseudo-hyperbolic metric, \( \chi \), on \( D \) is defined by \( \chi(z, w) = |z - w| / |1 - \bar{z}w| \). We recall two classical theorems from the theory of complex variables.

Pick's Theorem [1, p. 48]. Let \( f \in H^\infty \), and suppose \( f \) maps into \( D \). Then for any two points \( z, w \in D \) one has

\[
\chi(f(z), f(w)) \leq \chi(z, w).
\]

Lindelöf's Theorem [1, p. 76]. Let \( G \) be a region bounded by a simple closed curve \( \Gamma \), let \( p \in \Gamma \). Let \( f \) be continuous on \( (G \cup \Gamma) - \{p\} \), bounded and analytic on \( G \). If \( f(z) \) approaches the value \( a \) at \( p \) as \( z \) approaches \( p \) from either direction on \( \Gamma \), then \( f(z) \) approaches \( a \) as \( z \) tends to \( p \) through \( G \cup \Gamma - \{p\} \).

Recall that a sequence \( \{z_n\} \) in \( D \) is a Blaschke sequence if and only if \( \sum 1 - |z_n| \) converges. With this terminology and the above two theorems we prove

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Theorem 1. Let $K$ be a compact subset of $D \cup \Gamma$ such that $K \cap \Gamma = \{e^{i\theta}\}$. Then there exists a Blaschke sequence $\{z_n\}$ in $D$, $z_n \to e^{i\theta}$ with the property that whenever $f \in H^\infty$ and $f(z_n) \to 0$, then
$$\lim_{z \to e^{i\theta}: z \in K} f(z) = 0.$$  

Proof. We may assume $e^{i\theta} = 1$. Let $K'$ be the convex hull of $K \cup \overline{K}$, where $\overline{K}$ is the set of conjugates of the points of $K$. Let $\gamma'$ be the boundary of $K'$. By Lindelöf's Theorem to prove the present theorem it is enough to construct a Blaschke sequence $\{z_n\}$ on $\gamma'$ with the property that if $f \in H^\infty$ and $f(z_n) \to 0$, then $f(z) \to 0$ as $z \to 1$ on $\gamma'$. Now, the union of two Blaschke sequences tending to 1 is again a Blaschke sequence tending to 1 and $\gamma'$ is symmetric about the real axis. So it is sufficient to find a sequence $\{z_n\}$ on the part, $\gamma$, of $\gamma'$ terminating at 1 which lies above the real axis such that $f(z_n) \to 0$ implies $f(z) \to 0$ as $z \to 1$ along $\gamma$. We will use these additional properties of $\gamma$: $\gamma$ is convex hence, rectifiable; as $z$ proceeds along $\gamma$ to 1, $|z|$ and $\text{Re}(z)$ increase monotonely to 1. Denote the Euclidean arclength measured along $\gamma$ between $z$, $w \in \gamma$ by $\gamma(z, w)$.

With these preliminaries we proceed with the construction of $\{z_n\}$. Choose a point $w_1$ on $\gamma$. Let $w_2$ be that point on $\gamma$ satisfying $|w_2| > |w_1|$ and $\gamma(w_1, w_2) = 1 - |w_2|$. There is such a point since the length of $\gamma$ exceeds $1 - |w_1|$. This same procedure continued indefinitely from $w_1$ by induction yields a sequence $\{w_n\}$ on $\gamma$ such that $\gamma(w_n, w_{n+1}) = 1 - |w_{n+1}|$ and $|w_n| \to 1$. The latter follows since $\sum_{n=1}^{\infty} (1 - |w_{n+1}|) = \gamma(w_1, 1) < \infty$, and this also proves that $\{w_n\}$ is a Blaschke sequence. It is easy to find a sequence $\{N_n\}$ of integers with $N_n \to \infty$ while $\sum_{n=1}^{\infty} N_n (1 - |w_{n+1}|)$ is still convergent. Construct a new Blaschke sequence $\{z_n\}$ on $\gamma$ consisting for each $n$ of the points $w_n = w_{n,0}$, $w_{n,1}$, ..., $w_{n,N_n} = w_{n+1}$, where $\gamma(w_{n,j}, w_{n,j+1}) = N_n^{-1} \gamma(w_n, w_{n+1})$, $j = 0, \ldots, N_n - 1$. Let $z$ be any point on $\gamma$ between $w_{n,j}$ and $w_{n,j+1}$ inclusive. Then

$$N_n |z - w_{n,j+1}| \leq N_n \gamma(z, w_{n,j+1}) \leq N_n \gamma(w_n, w_{n,j+1}) = 1 - |w_{n+1}| \leq 1 - |w_{n,j+1}|.$$  

Thus,

$$\chi(z, w_{n,j+1}) \leq \frac{|z - w_{n,j+1}|}{1 - |w_{n,j+1}|} \leq \frac{1}{N_n}.$$  

Now, suppose $f \in H^\infty$, $f(z_n) \to 0$, and, without loss of generality, that $|f|$ is bounded by 1. Then, by Pick's Theorem and the last inequality
\[ \chi(f(z), f(w_{n, j+1})) \leq 1/N_n \to 0. \]

Since \( f(w_{n, j+1}) \to 0, f(z) \to 0 \) as \( z \to 1, z \in \gamma \). This completes the proof.

In particular, the Blaschke product with zeros \( \{z_n\} \) tends to zero as \( z \to e^{i\theta} \) through \( K \). By definition a function \( f \in \mathcal{H}^\infty \) has \( T_\mu \)-limit zero at \( 1, \mu \geq 1 \), in case \( f(z) \to 0 \) as \( z \to 1 \) through the sets

\[ K(m, \mu) = \{z: 1 - |z| \geq m(\arg z)^\mu, 0 < |z| < 1\} \]

for each \( m > 0 \). Thus, it is clear how Somadasa's Theorem may be obtained from Theorem 1.

Theorem 1 also points out that the approaches to 1 through the sets \( K(m, \mu), m > 0, \mu \geq 1 \), fall rather short of exhausting the different possible tangential approaches to 1. For let \( \gamma \) be a convex curve in \( D \cup \Gamma, \gamma \cap \Gamma = 1 \), which is symmetric about the real axis. Then, from the proof of Theorem 1, there is a Blaschke product, \( B(z) \), whose zeros are on \( \gamma \) and which tends to zero as \( z \to 1 \) inside the curve \( \gamma \). Furthermore, as is well known (e.g., see [2, p. 35]) \( B(z) \) tends to each number of modulus not exceeding 1 on some sequence \( \{z_n\} \) tending to 1. Each of the sequences must approach 1 more tangentially than the curve \( \gamma \). This situation persists independent of how large an order of contact at 1 is chosen for \( \gamma \).

References


University of California, Santa Barbara