ON THE EXISTENCE THEOREM FOR DIFFERENTIAL EQUATIONS

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In this note we show how to derive the fundamental existence theorem for ordinary differential equations as a corollary of the implicit function theorem in Banach spaces. The proof of smoothness with respect to initial conditions is considerably shorter than existing proofs (see, for example [3], [4], or [5]). Throughout, a dot (i.e., \( \dot{t} \)) denotes differentiation with respect to \( t \).

**Theorem.** Let \( U \) be an open set in a Banach space \( E \) and let \( f: \mathbb{R} \times U \rightarrow E \) be a \( C^r \) map \((r \geq 1)\). Then for each \( x_0 \in U \) there exists an open neighborhood \( V \) of \( x_0 \) in \( U \), an open interval \((-\epsilon, \epsilon)\) about 0 in \( \mathbb{R} \) and a map \( \phi: (-\epsilon, \epsilon) \times V \rightarrow U \) such that

1. \( \phi \) is \( C^r \);
2. \( \phi(0, x) = x \) for \( x \in V \);
3. \( \phi(t, x) = f(t, \phi(t, x)) \) for \((t, x) \in (-\epsilon, \epsilon) \times V \).

**Proof.** We suppose without loss of generality that \( x_0 \) is the origin of \( E \) and that \( U \) is an open ball with center \( x_0 \). Take \( U_0 \) to be the open ball whose center is \( x_0 \) and whose radius is half the radius of \( U \). Let \( I \) denote the closed interval \([-1, 1] \subseteq \mathbb{R} \). For \( p \) an integer \( \geq 0 \) let \( C^p(I, E) \) denote the Banach space of \( C^p \) maps from \( I \) to \( E \) (with the \( C^p \) topology), \( C^p_0(I, E) \) be the (closed) subspace of \( C^p(I, E) \) consisting of all \( \gamma \in C^p(I, E) \) with \( \gamma(0) = 0 \), and \( C^p_0(I, U_0) \) the set of all \( \gamma \in C^p_0(I, E) \) such that \( \gamma(I) \subseteq U_0 \). Note that \( C^p_0(I, U_0) \) is open in the Banach space \( C^p_0(I, E) \). \( D \) denotes the differentiation operator (see [4] or [5]) and \( D_j \) denotes partial differentiation with respect to the \( j \)th variable.

Let \( F: \mathbb{R} \times U_0 \times C^0_0(I, U_0) \rightarrow C^0_0(I, E) \) be the map defined by

\[
F(a, x, \gamma)(t) = \gamma(t) - af(at, x + \gamma(t))
\]

for \( a \in \mathbb{R}, x \in U_0, \gamma \in C^0_0(I, U_0) \) and \( t \in I \). One easily verifies that \( F \) is a \( C^1 \) map between Banach spaces. (This is an especially easy instance of the so-called omega theorem of [1]. Note that the map \( \gamma \rightarrow \gamma \) is continuous linear.) The partial derivative with respect to \( \gamma \) at the point \( a = 0, x = x_0, \gamma = 0 \) evaluated at the “tangent vector” \( \delta \in C^0_0(I, E) \) is given by

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1 I am indebted to R. Abraham for suggesting this to me.
\[ D_2 F(0, x_0, 0) \delta(t) = \dot{\delta}(t); \]

it is clearly a toplinear isomorphism. Since \( F(0, x_0, 0) = 0 \) we may apply the implicit function theorem [4, p. 265]. This yields an open neighborhood \((-2\epsilon, 2\epsilon) \times V \) of \((0, x_0)\) in \( \mathbb{R} \times U_0 \) and a \( C^1 \) map \( H: (-2\epsilon, 2\epsilon) \times V \rightarrow C^0_0(I, U_0) \) such that
\[
F(a, x, H(a, x)) = 0
\]
for \((a, x) \in (-2\epsilon, 2\epsilon) \times V \). We define \( \phi: (-\epsilon, \epsilon) \times V \rightarrow U \) by
\[
\phi(t, x) = H(\epsilon, x)(t/\epsilon) + x.
\]
\( \phi \) is \( C^1 \): this follows immediately from the fact that the evaluation map \( C^0_0(I, U_0) \times I \rightarrow U_0 \) is \( C^1 \) (see [1] or [2, p. 25]). \( \phi(0, x) = x \) since \( H(\epsilon, x) \in C^0_0(I, U_0) \). Finally, since
\[
\phi(t, x) - f(t, \phi(t, x)) = (1/\epsilon)F(\epsilon, x, H(\epsilon, x))(t/\epsilon) = 0
\]
it follows that \( \phi \) is the solution curve. We have proved the theorem in the case \( r = 1 \). The general case follows from the case \( r = 1 \) by an easy (and standard) induction argument.

**Bibliography**


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