ON GENERATING HEAVY POINTS WITH POSITIVE MATRICES

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Let S be a countable topological space. If S is indexed by the positive integers, then real functions on S may be considered as sequences and are subject to infinite matrix transformation. After Henriksen and Isbell in [**3**], a point s in S is called a *heavy point* if there exists a regular matrix which sums every bounded real-valued function on S continuous at s. It is clear that the notion of a heavy point is independent of the order chosen for S. Some properties of heavy points are given in [**3**]. For example, sequential limit points are heavy points and isolated points (in fact, points whose complements are C^*-embedded) are not heavy points. In addition Henriksen and Isbell observe that not all heavy points are sequential limit points. The latter phenomenon was the motivation for the present paper.

Corresponding to each positive regular matrix A we define a topology on the positive integers by taking neighborhoods of 1 to be sets whose characteristic functions are A-summable to 1. Denote the resulting space by N_A. The point 1 is a heavy point in N_A. Some basic properties of these spaces are examined. Also, we determine the class of matrices A for which 1 is not a sequential limit point in N_A.

Let $A=(a_{nk})$ be an infinite real matrix. The *A-transform* of the real sequence $x=\{x_k\}$ is the sequence $\{\sum_k a_{nk}x_k\}$ if these sums exist. Let c and m denote the Banach spaces of convergent sequences and bounded sequences, respectively, with $\|x\|=\sup_k|x_k|$. Let $c_A=\{x: Ax\in c\}$. The matrix A is said to *sum* the members of c_A or any subset of c_A. Define the functional $\lim_A$ on c_A by $\lim_A x=\lim Ax$. The matrix A is called *conservative* if $c\subset c_A$ and *regular* if in addition $\lim=\lim_A$ on c. It is well known that A is regular if and only if $\lim_n a_{nk}=0$ for all k, $\lim_n \sum_k a_{nk}=1$, and $\sup_n \sum_k |a_{nk}|<\infty$. The matrix A is called *positive* if $a_{nk}\geqq 0$ for all n, k.

For a set T of positive integers let $\chi(T)$ denote the characteristic function of T; $(\chi(T))_k=1$ for $k\in T$, 0 otherwise.

For a topological space S let $C^*(S)$ denote the Banach space of bounded real-valued continuous functions on S with $\|x\|=\sup\{|x(s)|: s\in S\}$. As observed in [**3**], if S is countable and completely regular, then S is 0-dimensional (i.e. the ring Γ of open and

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closed sets is a base for the topology of S) and the linear span of $\{\chi(U): U \in \Gamma\}$ is dense in $C^*(S)$.

Note that a conservative matrix A may be considered as a bounded linear transformation on m and that c is closed in m. It follows that $c_A \cap m$ is closed in m. Hence, if A sums $\{\chi(U): U \in \Gamma\}$, then it sums $C^*(S)$.

Let $\mathfrak{F}$ denote the set of finite subsets of $N \sim \{1\}$ where N is the positive integer. Let $\mathfrak{R}$ be a ring of subsets of N which includes $\mathfrak{F}$ and N. Let μ be a real finitely additive function on $\mathfrak{R}$, $\mu \geqq 0$, $\mu(N) = 1$, and $\mu = 0$ on $\mathfrak{F}$. Finally, let $\mathfrak{B}_\mu = \{U \cup \{1\}: U \in \mathfrak{R}, \mu(U) = 1\}$. Then $\mathfrak{B}_\mu \cup \mathfrak{F}$ is a base for a topology on N. The resulting space, call it N_μ, is Hausdorff, completely regular, and has at most one limit point. Actually, any countable space with these properties may be obtained as an N_μ space.

We shall be concerned here with the case in which μ is given by a positive regular matrix A. Let $\mathfrak{R} = \{U: \lim_A \chi(U) = 1 \text{ or } 0\}$, $\mu(U) = \lim_A \chi(U)$. Denote the resulting space by N_A.

EXAMPLES. If A is the identity matrix, then $n \to 1$ in N_A.

Let B be the $(C, 1)$ matrix, $b_{nk} = 1/n$ for $k \leqq n$, 0 otherwise. Then N_B is the space given by Appert in [**2**, p. 84]. Neighborhoods of the point 1 are sets with density 1. As pointed out by Sierpinski [**4**, p. 61] the space N_B is not first countable.

Let D be the matrix defined by $d_{nk} = 2^{-p}$ for $k = 2^{p-1}(2n-1)$, 0 otherwise. Then N_D is the space given in [**4**, p. 61]. Here 1 is a sequential limit point, but N_D is not first countable.

Corresponding to $A = (a_{nk})$, let $a_A^k = \sup_n a_{nk}$.

LEMMA. *Let* $S = \{n_k\}$ *be an infinite subset of* $N_A \sim \{1\}$.

(i) *If* $\lim_A \chi(S) = 0$, *then* $a_A^{n_k} \to 0$.

(ii) S *includes an infinite closed subset of* N_A *if and only if* $\liminf_k a_A^{n_k} = 0$.

PROOF. The proof of (i) is easy and is omitted.

Assume $\liminf_k a_A^{n_k} = 0$. Choose distinct positive integers r_k such that $\sum_k a_A^{n_{r_k}} < \infty$. Let $T = \{n_{r_k}: k = 1, 2, \cdots\}$. Then $\lim_A \chi(T) = \lim_n \sum_k a_{n,n_{r_k}} = 0$, since the series is uniformly convergent. Hence, T is closed in N_A by definition. The converse follows from (i).

THEOREM. (i) N_μ *is pseudo-finite* (*i.e. all compact sets are finite*) *if and only if* 1 *is not a sequential limit point.*

(ii) A *sums* $C^*(N_A)$; *hence,* 1 *is a heavy point in* N_A.

(iii) N_A *is not extremally disconnected.*

(iv) 1 *fails to be a sequential limit point in* N_A *if and only if* $a_A^k \to 0$.

(v) N_A *is first countable if and only if it is locally compact.*

PROOF. Part (i) is immediate.

It is clear that A sums $\chi(U)$ for every open and closed subset U of N_A, so part (ii) follows.

Actually, (iii) follows from (ii) and Theorem 3 of [**3**]. In our setting it is easier to choose a subset S of $N_A \sim \{1\}$ such that $\chi(S) \notin c_A$. (See [**5**, p. 54].) Then S and $T = N_A \sim (S \cup \{1\})$ are disjoint open sets, but $1 \in \bar{S} \cap \bar{T}$. Hence, N_A is not extremally disconnected.

Part (iv) follows from part (ii) of the lemma and the fact that 1 is a sequential limit point if and only if there exists an infinite subset of $N_A \sim \{1\}$ which includes no infinite closed subset of N_A.

If N_A is locally compact, then it is first countable by [**1**, p. 65, Theorem III], since every point of N_A is a G_δ. Conversely, let $\{U_k\}$ be a strictly decreasing basic sequence of neighborhoods of 1 in N_A. Now $\lim_A \chi(U_k \sim U_{k+1}) = 0$ for each k. If the set $U_k \sim U_{k+1}$ is frequently infinite, then using (i) of the lemma we may choose $n_k \in U_k \sim U_{k+1}$ for each k such that $\liminf_k a_A^{n_k} = 0$. By (ii) of the lemma, there exists a closed subset of $N_A \sim \{1\}$ which meets every U_k, contradicting the fact that $\{U_k\}$ is basic. If $U_k \sim U_{k+1}$ is finite, say for $k \geqq p$, then U_p is a compact neighborhood of 1 and N_A is locally compact. This completes the proof.

Note that half of part (v) fails for N_μ spaces in general.

It is clear that the compact N_A spaces are identical. Likewise, noncompact but locally compact N_A spaces are homeomorphic. However, N_A spaces which are not locally compact need not be homeomorphic. In fact we have

EXAMPLE. Let A be the $(C, 1)$ matrix. Let $B = (b_{nk})$ be defined by $b_{nk} = 1/n$ for $n(n-1)/2 < k \leqq n(n+1)/2$, 0 otherwise. According to (iv) of the theorem, 1 is not a sequential limit point of N_A or N_B. However, corresponding to each permutation π of N there exists a set S in N with $\lim_B \chi(S) \neq 1$ but which has density 1 with respect to π. Hence, N_A is not homeomorphic to N_B.

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