ON THE IDENTITY IN A MEASURE ALGEBRA

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1. Introduction. Let $G$ be a commutative locally compact topological semigroup, and let $M(G)$ denote the complex Banach algebra of all bounded regular Borel measures on $G$, where the product is defined by convolution. That is, if $\mu, \nu \in M(G)$ and $A$ is a Borel subset of $G$, then

$$(\mu * \nu)(A) = \int_G \int_G \varphi_A(xy) d\mu(x) d\nu(y),$$

where $\varphi_A$ denotes the characteristic function of $A$. A general treatment of the algebras $M(G)$ appears in [4] and also in the survey article [6]. Special algebras of this type are discussed in [3] and [2].

The set of discrete measures in $M(G)$ forms a subalgebra of $M(G)$, denoted by $\ell_1(G)$. Of course, if the topology on $G$ is discrete, then $\ell_1(G) = M(G)$. A comprehensive study of $\ell_1(G)$ was presented by Hewitt and Zuckerman in [1]. One of their results is that $\ell_1(G)$ contains an identity if and only if $G$ contains a finite set of relative units. The purpose of this note is to show that these same conditions on $G$ are necessary and sufficient for the existence of an identity in $M(G)$. It will follow that an identity for $M(G)$ must lie in $\ell_1(G)$.

The author is indebted to Professor A. Hudson for suggesting that the results in [5] could be used to simplify an earlier proof of Proposition 2.4.

2. Relative units. This section establishes certain properties of relative units. We begin with the definition.

Definition. A subset $U$ of $G$ is a set of relative units for $G$ if for every $x \in G$, there exists $u \in U$ such that $ux = x$. If no proper subset of $U$ is a set of relative units, then we call $U$ a minimal set of relative units for $G$.

Our attention shall be centered to a large extent on minimal sets of relative units for $G$. The semigroup consisting of the real numbers under the operation $xy = \max\{x, y\}$ furnishes an example of a semigroup that possesses sets of relative units, but fails to possess a minimal set of relative units. The latter fact is easily verified directly; however it is also a consequence of the following Proposition.

Proposition 2.1. Let $U$ be a set of relative units for $G$. Then the following are equivalent.

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(i) $U$ is minimal.

(ii) If $u, v \in U$ and $uv = v$, then $u = v$.

(iii) If $x \in G$, $u \in U$ and $ux = u$, then $x = u$.

**Proof.** (i) implies (ii). Let $U$ be minimal, and suppose that $u, v \in U$ with $uv = v$. Then whenever $vx = x$ obtains, we also have $ux = u(ux) = (uv)x = ux = x$. By the minimality of $U$ we must have $u = v$.

(ii) implies (iii). Suppose that $ux = u$ where $x \in G$ and $u \in U$. Choose $v \in U$ such that $vx = x$. Then $uv = (ux)v = u(vx) = ux = u$, and by (ii) we have $u = v$. Thus, $x = vx = ux = u$.

(iii) implies (i). If $U$ is not minimal, then we can find an element $u \in U$ such that $U \setminus \{u\}$ is a set of relative units for $G$. Thus, there exists $v \in U \setminus \{u\}$ such that $uv = v$. Since $u$ and $v$ are distinct, (iii) is not satisfied.

**Corollary 2.2.** If $U$ is a minimal set of relative units for $G$, then each element of $U$ is idempotent.

**Corollary 2.3.** If $U$ is a minimal set of relative units, then $U$ is contained in every set of relative units.

**Proof.** Let $U$ and $U'$ be sets of relative units for $G$ and suppose that $U$ is minimal. Given $u \in U$, there exists an element $u' \in U'$ such that $u'u = u$. For this $u'$ there exists an element $v \in U$ such that $vu' = u'$. Hence, $u = u'u = vu'u = vu$. By Proposition 2.1, it follows that $u = v$. Thus we conclude that $U \subseteq U'$.

**Proposition 2.4.** If $K$ is a compact set of relative units for $G$, then $K$ contains a minimal set of relative units.

**Proof.** Let $K_0 = \{u \in K : u^2 = u\}$. We shall show that $K_0$ is a set of relative units for $G$. Let $x$ be an arbitrary element in $G$. Choose $u_1 \in K$ such that $u_1x = x$. Now choose $u_2 \in K$ such that $u_2u_1 = u_1$. Then we also have $u_2x = x$. We can inductively determine a sequence $\{u_n\}$ in $K$ such that $u_{m+1}u_n = u_n$ for $m > n$ and $u_nx = x$. Since $K$ is compact, the sequence $\{u_n\}$ has a limit point $u \in K$. It is not difficult to show that $u^2 = u$ and that $ux = x$. It follows that $K_0$ is a set of relative units for $G$.

We define a partial order on $K_0$, in the usual way, by agreeing that $u \preceq u'$ if and only if $uu' = u$. A standard argument using Zorn’s lemma shows that each element of $K_0$ is contained in a maximal chain. By [5] every maximal chain is closed and has a maximal element.

Let $U$ be the set that contains the maximal element $u_L$ of every maximal chain $L$ of $K_0$. We claim that $U$ is the desired minimal set of relative units for $G$. Let $x \in G$; then there exists $u \in K_0$ such that
ux = x. Since u ∈ L for some L, and since uLu = u, we have uLx = x. Thus U is a set of relative units for G. It follows from Proposition 2.1 that U is minimal. For suppose that uLuL = uL; then the maximality conditions imply that uL = uL'.

3. The identity in \( M(G) \). We now apply the previous results to give conditions on G that are necessary and sufficient for the existence of an identity in \( M(G) \). Before we proceed, we mention the following notational conventions. The total variation of \( \mu \in M(G) \) is denoted by \( |\mu| \). For \( x \in G \), \( \varphi_x \) denotes the characteristic function of singleton \( \{x\} \), \( \delta_x \) denotes the unit point mass at \( x \), and \( V_x = \{y \in G : xy = x\} \).

**Lemma 3.1.** If \( M(G) \) contains an identity \( E \), then \( E(V_x) = 1 \) for all \( x \in G \).

**Proof.**

\[
1 = \delta_x(\{x\}) = (E \ast \delta_x)(\{x\}) = \int_G \int_G \varphi_x(yz) d\delta_x(y) dE(z)
\]

\[
= \int_G \varphi_x(xz) dE(z) = E(V_x).
\]

**Lemma 3.2.** If \( M(G) \) contains an identity \( E \), then \( G \) contains a compact set \( K \) of relative units.

**Proof.** Choose, by means of the regularity of \( E \), a compact subset \( K \) of \( G \) such that \( |E|(G) - |E|(K) < 1 \). Then from Lemma 3.1 we have \( 1 \leq |E|(V_x) = |E|(V_x \cap K) + |E|(V_x \setminus K) \). Since \( |E|(V_x \setminus K) \leq |E|(G \setminus K) < 1 \), we have \( |E|(V_x \cap K) > 0 \). Thus, for each \( x \in G \), \( V_x \cap K \) is not empty and we conclude that \( K \) is a set of relative units.

**Theorem 3.3.** \( M(G) \) contains an identity if and only if \( G \) contains a finite set of relative units.

**Proof.** If \( G \) contains a finite set of relative units, then a routine verification shows that the identity for \( l_1(G) \) presented in [1] is also an identity for \( M(G) \). Thus, if \( U = \{u_1, \ldots, u_n\} \) then we have

\[
E = \sum \{\delta_{u_i} : 1 \leq i \leq n\} - \sum \{\delta_{u_i u_j} : 1 \leq i < j \leq n\} + \cdots + (-1)^{n+1}\delta_{u_1 u_2 \cdots u_n}.
\]

Conversely, suppose that \( M(G) \) contains an identity \( E \). By Lemma 3.2, \( G \) contains a compact set of relative units. Thus, from Proposition 2.4, it follows that \( G \) contains a minimal set \( U \) of relative units. If \( u \in U \), then condition (iii) of Proposition 2.1 implies that \( V_u = \{u\} \).
Thus by Lemma 3.1, $E(\{u\}) = 1$. Finally, the boundedness of $E$ implies that $U$ is finite.

REFERENCES


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