ON THE EXISTENCE OF RIGID COMPACT
ORDERED SPACES

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1. It is easily seen that every compact ordered space with
infinitely many points which has a countable base admits continuously
many autohomeomorphisms. For, if there are countably many iso-
lated points, the assertion is obvious. In the other case the assertion
follows from the fact that there is either a separable connected sub-
space which is consequently homeomorphic to an interval of the real
numbers or the space is zero-dimensional and so is homeomorphic to
the Cantor set (possibly except for finitely many isolated points).
Jónsson [1] and Rieger [2] each give an example of an infinite zero-
dimensional compact ordered space $S$, which is rigid, i.e. a space such
that the only homeomorphism of $S$ onto $S$ is the identity mapping.
However, the weight of the constructed space $S$ is very large. (Here
weight means the minimal cardinality of an open base.) The purpose
of this note is to show the existence of (a family of $2^c$) rigid zero-
dimensional compact ordered spaces each having continuous weight
and continuous power. (Here $c$ denotes the cardinal of the set of real
numbers.)

2. If $N$ is a set and $f$ is a map of a subset $S$ of $N$ into $N$, then for
every $M \subseteq N$ we write $fM$ instead of $f[S \cap M]$.
In de Groot [1] the following concepts were introduced:
(a). If $f$ is a map of $S \subseteq N$ into $N$, then $f$ is called a displacement of
order $m$, if there exists a subset $V$ of $N$ such that
$$V \cap fV = \emptyset, \quad \lvert fV \rvert = m,$$
whereas for no $n > m$ there is a subset $W$ of $N$, such that
$$W \cap fW = \emptyset, \quad \lvert fW \rvert = n.$$ (b). If $N$ is a topological space, then $f$ is called a continuous displace-
ment of $S \subseteq N$, if $f$ is a continuous map of $S$ into $N$ and $f$ is a displace-
ment of order $c$.
One easily proves the following generalization of [1, Lemma 2].

**Lemma.** Let $P$ be a separable metric space. Let $Q$ be a subset of $P$ of
which every point is a point of condensation in $Q$.

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If \( \psi : Q \rightarrow P \) is a map (\( \neq \) identity) such that every point of \( \psi[Q] \) is again a point of condensation in \( \psi[Q] \), then \( \psi \) is a continuous displacement in \( P \).

Strengthening in a trivial way the result of \([1, \text{Theorem 1}] \), for the case that \( M = \) the unit interval \( I \) of the reals and \( \{K_\theta \} \) is the system of all uncountable compact subsets of \( I \) (cf. \([1, \text{Theorem 2}] \)), we obtain the following theorem.

**Theorem A.** There exists a family \( \{ F_\gamma \} \) of \( 2^c \) zero-dimensional subsets of \( I \), each of power \( c \), such that

- (i) every subinterval of \( I \) contains \( c \) points of each \( F_\gamma \) and of each \( I \setminus F_\gamma \) (in particular \( \overline{F_\gamma} = I \) for all \( \gamma \)).
- (ii) \( |F_\gamma \setminus F_{\gamma'}| = c \) for all \( \gamma, \gamma' \) with \( \gamma \neq \gamma' \).
- (iii) \( |F_\gamma \setminus F_{\gamma'}| = c \) for all \( \gamma, \gamma' \) and all continuous displacements \( \phi \) of \( F_\gamma \) in \( I \).

From this theorem it follows \([1, \text{Theorem 2}] \), that there exists neither a proper autohomeomorphism of any \( F_\gamma \), nor a homeomorphism of any \( F_\gamma \) onto some other \( F_{\gamma'} \). (For, such a map would be a continuous displacement of \( F_\gamma \) (cf. \([1, \text{Lemma 2}] \)), which is impossible according to Theorem A(iii).)

3. Define the set

\[
S_\gamma = F_\gamma \cup \{(b, 0), (b, 1)\}_{b \in I \setminus F_\gamma}
\]

and introduce an order in it in the natural way (i.e. if \( \bar{p} = p \) when \( p \in F_\gamma \) and \( \bar{b} = b \) when \( p = (b, 0) \) or \( p = (b, 1) \), then \( p < q \) in \( S \) if \( \bar{p} < \bar{q} \) in \( I \); and moreover \( (b, 0) < (b, 1) \) for every \( b \in I \setminus F_\gamma \)). It is clear that \( S_\gamma \) (supplied with its interval topology) is a separable zero-dimensional compact space of weight \( c \).

Every subspace of \( S_\gamma \) which contains \( c \) points of \( S_\gamma \setminus F_\gamma \) has weight \( c \). And \( F_\gamma \)-as-a-subspace-of-\( I \) is homeomorphic to \( F_\gamma \)-as-a-subspace-of-\( S_\gamma \).

**Theorem B.** \( S_\gamma \) is rigid for every \( \gamma \).

\( S_\gamma \) is not homeomorphic to \( S_{\gamma'} \), if \( \gamma \neq \gamma' \).

**Proof.** Let \( \phi_\gamma \) be the continuous map of \( S_\gamma \) onto \( I \) which is defined by

\[
\phi_\gamma(a) = a, \quad \text{if} \ a \in F_\gamma,
\]

\[
\phi_\gamma((b, 0)) = \phi_\gamma((b, 1)) = b, \quad \text{if} \ b \in I \setminus F_\gamma.
\]

Now suppose that \( f_\gamma \) is a proper autohomeomorphism of \( S_\gamma \), and that
$f_{\gamma'}$ is a homeomorphism of $S_\gamma$ onto $S_{\gamma'}$ (\(\gamma \neq \gamma'\)). Then it is easily checked (see Lemma) that

$$\psi_\gamma = (\phi_\gamma \cdot f_\gamma) \mid F_\gamma \quad \text{and} \quad \psi_{\gamma'} = (\phi_{\gamma'} \cdot f_{\gamma'}) \mid F_\gamma$$

are continuous displacements of $F_\gamma$ in $I$. Consequently (Theorem A(iii)) $I \setminus F_\gamma$ contains $c$ points of $\psi_\gamma F_\gamma$ and $I \setminus F_\gamma$ contains $c$ points of $\psi_{\gamma'} F_{\gamma'}$, and so $S_\gamma \setminus F_\gamma$ contains $c$ points of $f_\gamma F_\gamma$ and $S_{\gamma'} \setminus F_{\gamma'}$ contains $c$ points of $f_{\gamma'} F_{\gamma'}$. However, this is a contradiction, since $f_\gamma F_\gamma$ and $f_{\gamma'} F_{\gamma'}$ then would have weight $c$, whereas $F_\gamma$ has weight $\aleph_0$.

**Corollary.** If $B(X)$ denotes the Boolean algebra of the open-and-closed subsets of the zero-dimensional compact space $X$, then \(\{B(S_\gamma)\}_{\gamma}\) is a family of $2^c$ Boolean algebras, each of power $c$, without proper automorphisms.

**References**


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