ON CLOSING MANIFOLDS FIBERED OVER SURFACES

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1. **Introduction.** Given a manifold it is sometimes possible to make identifications on its boundary to obtain a closed manifold. For example any closed n-manifold may be obtained from an n-cell in this manner [**2**]. In order to investigate which closed manifolds may be obtained from a class of manifolds with boundary we assume the additional structure of a singular fibering [**12**]. Analysis of the fundamental group then gives necessary conditions for two closings to be homeomorphic.

2. **Singular fiberings.** Let $\overline{M}$ be a locally trivial fiber space with fiber F, a connected $(n-2)$-manifold, and base $\overline{B}$, a sphere with $g \geqq 0$ handles (or $k>0$ crosscaps) and r boundary components. Further assume $\overline{M}$ can be obtained as follows. Let B be a sphere with $g \geqq 0$ handles (k crosscaps). Cutting B along a set of loops based at v_1 gives a $4g$-gon ($2k$-gon) P with sides $A_1B_1A_1^{-1}B_1^{-1} \cdots B_g^{-1}$ ($C_1C_1' C_2 \cdots C_kC_k'$) to be identified in pairs. Removing an open disk D_0 around v_1 gives a polygon $\overline{P}$ which is P with a sector of a disk removed from each vertex. $\overline{P} \times F$ is an n-manifold on which we shall make some identifications.

Let $\chi: \pi_1(B, v_0) \to$ Automorphisms of $\pi_1(F)$ be a homomorphism where v_0 is the center of P and the automorphisms act on the right of $\pi_1(F)$. Let x, x' be points on the straight edges of $\overline{P}$ which will be identified in B and let α be the path formed by the straight lines v_0 to x and x' to v_0 which is a loop in B based at v_0. Then choose a base point preserving homeomorphism $x \times F \to x' \times F$ which induces $\chi([\alpha]): \pi_1(F) \to \pi_1(F)$. Identifying pairs of fibers over the straight edges of $\overline{P}$ by these homeomorphisms gives a manifold $\overline{M}_0$ with a boundary $\partial D_0 \times F$. Now suppose $\partial D_0 \times F$ is trivially fibered by circles ω such that $[\omega] = d_0 + t \in \pi_1(\partial D_0 \times F)$ where d_0 generates $\pi_1(\partial D_0)$ and $t \in \pi_1(F)$. By identifying each ω on $\partial D_0 \times F$ to a point we obtain a closed n-manifold M_0 [**1**, p. 229] fibered by F over B. The homomorphism χ is its characteristic and $t \in \pi_1(F)$ is the obstruction to a cross section. By removing the fibers over open disks D_j, $j = 1, \cdots, r$, in B we obtain $\overline{M}$ with r boundary components $\partial D_j \times F$.

Suppose $\partial D_j \times F$ is trivially fibered by circles ω_j such that $[\omega_j] = n_j d_j + u_j \in \pi_1(\partial D_j \times F)$ where n_j is a nonzero integer, d_j a generator of $\pi_1(\partial D_j)$ and $u_j \neq 1 \in \pi_1(F)$. By identifying each ω_j to a point $\overline{M}$ is

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transformed into another n-manifold with one less boundary component. The $\partial D_j \times F$ has been replaced by a space F_j homeomorphic to F. F_j is a singular fiber of type n_j [12]. Note F_j will have a trivial product neighborhood if and only if $n_j = \pm 1$. Replacing all boundary components of $\overline{M}$ by singular fibers in this manner we obtain a closed manifold M which we say is singularly fibered by F over a surface.

For example, a locally-trivial fibering of a 4-manifold by a closed surface is fiber preserving homeomorphic to such an $\overline{M}$. If F is taken to be a circle, then the singularly fibered M are the Seifert manifolds [11]. Recently it has been shown by Orlik, Vogt and Zieschang [10] that for Seifert manifolds, with a few exceptions, the fundamental group determines the manifold up to a fiber preserving homeomorphism. In that direction we have the following results for a more general fiber.

3. **Fundamental groups and singular fiberings.**

THEOREM 1. *Let M be singularly fibered over a closed surface by a path connected fiber F with $\pi_1(F)$ abelian. If the base space is a sphere, torus, or projective plane also assume $r-2 > \sum 1/n_j$, $r \geqq 1$, or $r \geqq 2$ respectively, where r is the number of singular fibers, and the types are n_j, $j=1, \cdots, r$. Then $\pi_1(M)$ determines the base space, the number of singularities and the types of singularities.*

THEOREM 2. *Let M be singularly fibered by F over a closed surface B. If $\pi_1(F)$ can be finitely presented and if F has first Betti number greater than one then $\pi_1(M)$ is infinite.*

To prove these results we need a presentation of $\pi_1(M)$. Suppose $\pi_1(F)$ is presented by $(s_l \mid w_m)$, B has $g \geqq 0$ handles (or $k > 0$ crosscaps), and M has characteristic $\chi: \pi_1(B) \to \operatorname{Aut} \pi_1(F)$, obstruction term $t \in \pi_1(F)$ and singular fibers F_j produced by circles ω_j with $[\omega_j] = n_j d_j + u_j \in \pi_1(\partial D_j \times F)$. Then a straightforward application of Van Kampen's theorem shows that $\pi_1(M)$ is presented by generators: a_i, b_i, (or c_i), d_j, s_l; $i=1, \cdots, g$ (or k), $j=1, \cdots, r$, and relations: $\prod [a_i, b_i] d_r^{-1} \cdots d_1^{-1} t$ (or $\prod c_i^2 d_r^{-1} \cdots d_1^{-1} t$), w_m, $d^{n_j} u_j$, $[d_j, s_l]$, and $x^{-1} s_l x = s_l \chi(x)$ where $x = a_i$, b_i (or c_i).

LEMMA 1. *Let*

$$K_1(g; n_1, \cdots, n_r) = (a_i, b_i, d_j \mid \prod [a_i, b_i] d_1 \cdots d_r, d_j^{n_j}) \qquad g > 0,\ r > 0$$

$$K_2(k; n_1, \cdots, n_r) = (c_i, d_j \mid \prod c_i^2 d_1 \cdots d_r, d_j^{n_j}) \qquad k > 0,\ r > 0 \quad \textit{and} \quad r \geqq 2 \quad \textit{if } k = 1,$$

$$K_3(g; -) = (a_i, b_i \mid \prod [a_i, b_i]), \quad g \geqq 2,$$

$$K_4(k; -) = (c_i \mid \prod c_i^2), \quad k \geqq 2,$$

$$K_5(-; n_1, \cdots, n_r) = (d_j \mid d_1 \cdots d_r, d_j^{n_j}), \quad r - 2 > \sum 1/n_j.$$

Then all of these groups are nonisomorphic in pairs, that is if $K_i(x; p_1 \cdots p_r) \approx K_j(y; q_1, \cdots, q_s)$ *then* $i=j$, $x=y$, $r=s$, *and* $q_1, \cdots, q_s$ *is a permutation of* $p_1, \cdots, p_r$.

Proof. These follow from known results, e.g. [**3**, p. 203], [**6**, p. 239], [**8**, p. 55], [**9**, pp. 12–42], and [**13**].

Lemma 2. *A non-Euclidean crystallographic group of the form* K_i *in Lemma* 1 *contains no nontrivial normal abelian subgroup.*

Proof. Let K be of the form K_1, K_3, or K_5. Then K has a faithful representation as a Fuchsian group of the first kind [**6**, p. 239], so we may consider it as a group of linear transformations of the upper half plane [**7**, p. 1]. Therefore any abelian subgroup $A \subset K$ is cyclic [**7**, p. 15]. Now K has a trivial center since it is the amalgamated free product of two groups, one of which has a trivial center [**5**, p. 32]. In particular, A is not contained in the center of K. Thus there exist elements $g \in K$, $s \in A$ which do not commute. By [**7**, p. 9], two elements commute if and only if they have the same fixed points, so g and s have different fixed points. Thus gsg^{-1} has fixed points different than those of s so gsg^{-1} is not in A and A is not normal.

Suppose K is of the form K_2 or K_4 and let A be a normal abelian subgroup. K contains a Fuchsian group F as a normal subgroup of index 2. Then $A \cap F$ is a normal abelian subgroup of F and by the argument above $A \cap F = 1$. Thus A is presented by $(a \mid a^2 = 1)$. Since A is normal we have $gag^{-1} = a$ for all $g \in K$, that is a is in the center of K. But if $k \geqq 2$, K is the free product of $(c_1 \mid -)$ and $(c_2, \cdots, c_k, d_1, \cdots, d_r \mid d_j^{n_j})$ (or $(c_2, \cdots, c_k \mid -)$) amalgamated along subgroups isomorphic to Z generated by c_1^{-2} and $c_2^2 \cdots c_k^2 d_1 \cdots d_r$ (or $c_2^2 \cdots c_k^2$). Thus the center of K is torsion free. If $K = (c, d_1, \cdots, d_r \mid c^2 d_1 \cdots d_r, d_j^{n_j})$ $r \geqq 2$, then K is the free product of $(c \mid -)$ and $(d_1, \cdots, d_r \mid d_j^{n_j})$ amalgamated along subgroups isomorphic to Z generated by c^{-2} and $d_1 \cdots d_r$. Here K has no center since $(d_1, \cdots, d_r \mid d_j^{n_j})$ has no center.

Proof of Theorem 1. Let S be the subgroup of $\pi_1(M)$ consisting of all words in the s_l generators. The presentation of $\pi_1(M)$ shows S is a normal abelian subgroup. S is characterized as the unique maximal normal abelian subgroup of $\pi_1(M)$. For suppose T is another normal abelian subgroup of $\pi_1(M)$ and $S \cap T \neq T$. Then TS is a normal subgroup containing S and $TS/S \approx T/S \cap T$ is a nontrivial normal abelian subgroup of $\pi_1(M)/S$. But $\pi_1(M)/S$ is a group of the type listed in Lemma 2 so it has no nontrivial normal abelian subgroup. Thus S is a characteristic subgroup of $\pi_1(M)$ and $\pi_1(M)/S$ is independent of the presentation of $\pi_1(M)$. By Lemma 1, the B, r, and $n_1, \cdots, n_r$ are therefore determined.

PROOF OF THEOREM 2. $\pi_1(M)$ finite implies $\pi_1(B)$ must be finite so either $B=S^2$ or $B=P^2$. If $B=S^2$ then $\pi_1(M)=(s_l, d_j \mid d_j^{n_j}u_j, \prod d_j t, w_m, [s_l, d_j])$ with $l=1, \cdots, p$, $m=1, \cdots, q$, $j=1, \cdots, r$. The Jacobian of $\pi_1(M)$ evaluated at the trivializer [3, pp. 198–204] has $p+r$ columns and $r+1+q$ nonzero rows. Since the first Betti number of F is at least 2, the q rows corresponding to the relations w_m have rank at most $p-2$. Thus the Jacobian has rank at most $p+r-1$ and the nullity is at least one. Hence M has Betti number at least 1 and $\pi_1(M)$ is infinite.

If $B=P^2$ then $\pi_1(M)=(c, s_l, d_j \mid c^2 d_r^{-1} \cdots d_1^{-1}t, d_j^{n_j}u_j, w_m, [s_l, d_j], c^{-1}s_l c = s_l\alpha)$ where $l=1, \cdots, p$, $j=1, \cdots, r$, $m=1, \cdots, q$, $\alpha \in \operatorname{Aut} \pi_1(F)$ is such that $\alpha^2=\mathrm{id}$, and $u_j, t \in \pi_1(F)$. Adding the relations $s_l=1$ we obtain $K=(c, d_j \mid d_j^{n_j}, c^2 d_r^{-1} \cdots d_1^{-1})$. If $r \geqq 2$, K is infinite since it is an amalgamated free product. Hence we need only consider $\pi_1(M)$ with $B=P^2$ and $r=0$ or 1.

Consider $\pi=(c, d, s_l \mid c^2 d^{-1}t, d^n u, w_m, [s_l, d], c^{-1}s_l c=s_l\alpha)$ where $t, u \in \pi_1(F)$, $\alpha \in \operatorname{Aut} \pi_1(F)$ with $\alpha^2=\mathrm{id}$. We shall show π is a normal extension of $N=(s_l \mid w_m, [s_l, t])$ by $(c \mid c^{2n})$ [4, p. 225]. Since $\alpha^2=\mathrm{id}$ we can present π by $(c, s_l \mid c^{2n}t^n u, w_m, [s_l, tc^2], c^{-1}s_l c=s_l\alpha)$. Defining $s_l^c=c^{-1}s_l c$ gives an automorphism $\bar{\alpha}$ of N such that $\bar{\alpha}$ fixes $(t^n u)^{-1}$ and its $2n$th iterate is conjugation by $(t^n u)^{-1}$. Then defining the factor sets $(c^i, c^j)=1$ if $i+j \leqq 2n-1$ and $(c^k, c^j)=(t^n u)^{-1}$ if $i+j \geqq 2n$ specifies a group extension G which is isomorphic to π and contains the infinite subgroup N. For the case $r=0$ we set $n=1$, $u=1 \in \pi_1(F)$ and omit the relations $[s_l, d]$ and $[s_l, t]$ to obtain $\pi_1(F)$ as a subgroup of π. Note the requirement that first Betti number $\geqq 2$ cannot be dropped. Seifert shows with F a circle we can singularly fiber a three sphere [11, p. 206].

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