INTEGRAL REPRESENTATIONS AND REFINEMENT-UNBOUNDEDNESS

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1. **Introduction.** Suppose U is a set, $\boldsymbol{F}$ is a field of subsets of U, $\mathfrak{p}$ is the set of all real-valued functions defined on $\boldsymbol{F}$, $\mathfrak{p}_A$ is the set of all bounded and finitely additive elements of $\mathfrak{p}$, $\mathfrak{p}^+$ is the set of all non-negative-valued elements of $\mathfrak{p}$, and $\mathfrak{p}_A^+ = \mathfrak{p}_A \cap \mathfrak{p}^+$.

Suppose μ is in $\mathfrak{p}_A^+$.

DEFINITION. If $\mathfrak{M}$ is a number set and ξ is in $\mathfrak{p}_A$, then the statement that ξ is μ-dense in $\mathfrak{M}$ means that if V is in $\boldsymbol{F}$ and $0<c$, then there is a subdivision $\mathfrak{E}$ of V and a function $\mathfrak{n}$ from $\mathfrak{E}$ into $\mathfrak{M}$ such that

$$\sum_{\mathfrak{E}} \left| \xi(I) - \mathfrak{n}(I)\mu(I) \right| < c.$$

We prove the following integral representation theorem (§3):

THEOREM 3.1. *If $\mathfrak{M}$ is a bounded number set and ξ is an element of $\mathfrak{p}_A$ which is μ-dense in $\mathfrak{M}$, then there is a function $\boldsymbol{\theta}$ from $\boldsymbol{F}$* into $\overline{\mathfrak{M}}$ (*i.e.*, $\mathfrak{M}$ *plus its closure*) *such that if V is in $\boldsymbol{F}$, then the integral* (§2)

$$\int_V \boldsymbol{\theta}(I)\mu(I)$$

exists and is $\xi(V)$.

The question naturally arises as to necessary and sufficient conditions under which, in the statement of Theorem 3.1, $\overline{\mathfrak{M}}$ may be replaced by $\mathfrak{M}$. By considering the previously defined [**1**] notion of refinement-unboundedness (see [**1**] or §4 of this paper), we obtain the following characterization theorem (§4):

THEOREM 4.1. *The following three statements are equivalent:*

(1) *If $\mathfrak{M}$ is a bounded number set and ξ is an element of $\mathfrak{p}_A$ which is μ-dense in $\mathfrak{M}$, then there is a function $\boldsymbol{\phi}$ from $\boldsymbol{F}$ into $\mathfrak{M}$ such that if V is in $\boldsymbol{F}$, then $\int_V \boldsymbol{\phi}(I)\mu(I)$ exists and is $\xi(V)$.*

(2) *If $\mathfrak{M}$ is a bounded number set and $\boldsymbol{\theta}$ is a function from $\boldsymbol{F}$ into $\overline{\mathfrak{M}}$ and $\int_U \boldsymbol{\theta}(I)\mu(I)$ exists, then there is a function $\boldsymbol{\phi}$ from $\boldsymbol{F}$ into $\mathfrak{M}$ such that if V is in $\boldsymbol{F}$, then $\int_V \boldsymbol{\phi}(I)\mu(I)$ exists and is $\int_V \boldsymbol{\theta}(I)\mu(I)$.*

(3) *There is a μ-refinement-unbounded* (§4) *element of $\mathfrak{p}^+$.*

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2. **Preliminary theorems and definitions.** If V is in $\boldsymbol{F}$, then the statement that $\mathfrak{D}$ is a subdivision of V means that $\mathfrak{D}$ is a finite collection of mutually exclusive sets of $\boldsymbol{F}$ whose union is V.

If $\mathfrak{D}$ is a subdivision of a set V of $\boldsymbol{F}$, then the statement that $\mathfrak{E}$ is a refinement of $\mathfrak{D}$ means that $\mathfrak{E}$ is a subdivision of V, every set of which is a subset of some set of $\mathfrak{D}$.

Throughout this paper all integrals considered will be Hellinger [**3**] type limits (i.e. for refinements of subdivisions) of the appropriate sums.

Suppose $\boldsymbol{\alpha}$ is in $\mathfrak{p}$.

Suppose $\mathfrak{G}$ is a subdivision of U. If V is in $\boldsymbol{F}$, then the statement that $\boldsymbol{\alpha}$ is $\sum$-bounded on V with respect to $\mathfrak{G}$ means that if $\mathfrak{N} = \{z \mid z = \sum_{\mathfrak{E}} \boldsymbol{\alpha}(I),\ \mathfrak{E}$ a subdivision of V and a subset of a refinement of $\mathfrak{G}\}$, then $-\infty < \mathbf{s}_*(\boldsymbol{\alpha})(V) = \inf \mathfrak{N} \leqq \sup \mathfrak{N} = \mathbf{s}^*(\boldsymbol{\alpha})(V) < \infty$. We adopt the convention that throughout this paper, as in the preceding definition, $\mathbf{s}^*$ and $\mathbf{s}_*$ will be understood to be defined in terms of the last mentioned subdivision in the discussion at hand with respect to which the functions under consideration are $\sum$-bounded on U. We see that $\boldsymbol{\alpha}$ is $\sum$-bounded on U with respect to $\mathfrak{G}$ iff for each V in $\boldsymbol{F}$, $\boldsymbol{\alpha}$ is $\sum$-bounded on V with respect to $\mathfrak{G}$, in which case, if V is in $\boldsymbol{F}$ and $\mathfrak{E}$ is a refinement of each of the subdivisions $\mathfrak{D}$ and $\mathfrak{D}'$ of V, then

$$\sum_{\mathfrak{D}} \mathbf{s}_*(\boldsymbol{\alpha})(I) \leqq \sum_{\mathfrak{E}} \mathbf{s}_*(\boldsymbol{\alpha})(I) \leqq \sum_{\mathfrak{E}} \mathbf{s}^*(\boldsymbol{\alpha})(I) \leqq \sum_{\mathfrak{D}'} \mathbf{s}^*(\boldsymbol{\alpha})(I),$$

so that we have the existence and following relationship of the following integrals:

$$\int_V \mathbf{s}_*(\boldsymbol{\alpha})(I) \leqq \int_V \mathbf{s}^*(\boldsymbol{\alpha})(I),$$

and we see that $\int_V \boldsymbol{\alpha}(I)$ exists iff $\int_V \mathbf{s}_*(\boldsymbol{\alpha})(I) = \int_V \mathbf{s}^*(\boldsymbol{\alpha})(I)$, in which case $\int_V \mathbf{s}_*(\boldsymbol{\alpha})(I) = \int_V \boldsymbol{\alpha}(I) = \int_V \mathbf{s}^*(\boldsymbol{\alpha})(I)$.

We observe that $\int_U \boldsymbol{\alpha}(I)$ exists iff for each V in $\boldsymbol{F}$, $\int_V \boldsymbol{\alpha}(I)$ exists. We take for granted the linearity and field-wise-additive properties of our integrals.

We state without proof a theorem of Kolmogoroff [**4**]:

THEOREM 2.K.1. *If $\int_U \boldsymbol{\alpha}(I)$ exists, then* $\int_U \left| \boldsymbol{\alpha}(I) - \int_I \boldsymbol{\alpha}(J) \right| = 0$.

Suppose each of $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ is a number sequence. We have the following two inequalities:

$$\min\{a_1, \cdots, a_n\} + \min\{b_1, \cdots, b_n\} \leqq \min\{a_1 + b_1, \cdots, a_n + b_n\},$$

$$\left| \min\{a_1, \cdots, a_n\} - \min\{b_1, \cdots, b_n\} \right| \leqq \sum_{i=1}^{n} | a_i - b_i | .$$

The first of the above inequalities implies that if $\{\beta_i\}_{i=1}^n$ is a sequence of elements of $\mathfrak{p}_A^+$ and $\mathfrak{E}$ is a refinement of a subdivision $\mathfrak{D}$ of a set V of $\boldsymbol{F}$, then

$$0 \leqq \sum_{\mathfrak{E}} \min\{\beta_1(I), \cdots, \beta_n(I)\} \leqq \sum_{\mathfrak{D}} \min\{\beta_1(I), \cdots, \beta_n(I)\},$$

so that

$$\int_V \min\{\beta_1(I), \cdots, \beta_n(I)\}$$

exists.

An immediate consequence of Theorem 2.K.1 and the second of the above inequalities is the following corollary which we state without proof:

COROLLARY 2.K.1. *If $\{\beta_i\}_{i=1}^n$ is a sequence of elements of $\mathfrak{p}^+$ such that $\int_U \beta_i(I)$ exists for $i=1, \cdots, n$, then*

$$\int_U \left| \min\{\beta_1(I), \cdots, \beta_n(I)\} - \min\left\{\int_I \beta_1(J), \cdots, \int_I \beta_n(J)\right\}\right| = 0,$$

so that if V is in $\boldsymbol{F}$, then

$$\int_V \min\{\beta_1(I), \cdots, \beta_n(I)\}$$

exists and is

$$\int_V \min\left\{\int_I \beta_1(I), \cdots, \int_I \beta_n(I)\right\}.$$

If in subsequent statements, the existence of a given integral or its equivalence to a given integral is an immediate consequence of the statements of this section, the integral need only be written and the proof of existence or equivalence left to the reader.

3. **The representation theorem.** In this section we prove Theorem 3.1, as stated in the introduction.

PROOF OF THEOREM 3.1. We see that there is a function β from $\boldsymbol{F}$ into $\overline{\mathfrak{M}}$ such that if V is in $\boldsymbol{F}$, then

$$\left| \xi(V) - \beta(V)\mu(V) \right| = \inf\{z \mid z = \left| \xi(V) - x\mu(V) \right|, x \text{ in } \mathfrak{M}\}.$$

Suppose $0<c$. There is a finite subset $\{a_1, \cdots, a_n\}$ of $\mathfrak{M}$ such that if x is in $\overline{\mathfrak{M}}$, then $\min\{|x-a_1|, \cdots, |x-a_n|\}<c/[4(\mu(U)+1)]$.

We see that $\int_U \min\{|\xi(I)-a_1\mu(I)|, \cdots, |\xi(I)-a_n\mu(I)|\}$ exists, since $\int_U|\xi(I)-a_i\mu(I)|$ exists for $i=1, \cdots, n$.

There is a function γ from $\boldsymbol{F}$ into $\{a_1, \cdots, a_n\}$ such that if I is in $\boldsymbol{F}$, then

$$|\xi(I) - \gamma(I)\mu(I)| = \min\{|\xi(I) - a_1\mu(I)|, \cdots, |\xi(I) - a_n\mu(I)|\}.$$

There is a subdivision $\mathfrak{D}$ of U such that if $\mathfrak{E}$ is a refinement of $\mathfrak{D}$, then

$$\left| \int_U | \xi(I) - \gamma(I)\mu(I) | - \sum_{\mathfrak{E}} | \xi(I) - \gamma(I)\mu(I) | \right| < c/4.$$

For each I in $\mathfrak{D}$ there is a subdivision $\mathfrak{E}_I$ of I and a function $\mathfrak{n}_I$ from $\mathfrak{E}_I$ into $\mathfrak{M}$ such that $\sum_{\mathfrak{E}_I} |\xi(J)-\mathfrak{n}_I(J)\mu(J)| <c/4N$, where N is the number of elements of $\mathfrak{D}$, and there is a function λ_I from $\mathfrak{E}_I$ into $\{a_1, \cdots, a_n\}$ such that for each J in $\mathfrak{E}_I$,

$$| \lambda_I(J) - \mathfrak{n}_I(J) | < c/[4(\mu(U) + 1)].$$

Therefore

$$\begin{aligned}
\int_U | \xi(J) - \gamma(J)\mu(J) | &< c/4 + \sum_{\mathfrak{D}} \sum_{\mathfrak{E}_I} | \xi(J) - \gamma(J)\mu(J) | \\
&\leqq c/4 + \sum_{\mathfrak{D}} \sum_{\mathfrak{E}_I} | \xi(J) - \lambda_I(J)\mu(J) | \\
&\leqq c/4 + \sum_{\mathfrak{D}} \sum_{\mathfrak{E}_I} | \xi(J) - \mathfrak{n}_I(J)\mu(J) | \\
&\quad + \sum_{\mathfrak{D}} \sum_{\mathfrak{E}_I} | \mathfrak{n}_I(J) - \lambda_I(J)| \mu(J) \\
&< c/4 + N(c/4N) + \{c/[4(\mu(U) + 1)]\}\mu(U) < 3c/4.
\end{aligned}$$

For each I in $\mathfrak{D}$, there is a subdivision $\mathfrak{E}_I'$ of I such that $0\leqq \mathbf{s}^*(|\xi-\beta\mu|)(I)-\sum_{\mathfrak{E}_I'}|\xi(J)-\beta(J)\mu(J)| <c/16N$.

Now

$$\begin{aligned}
\int_U \mathbf{s}^*(| \xi - \beta\mu |)(I) &\leqq \sum_{\mathfrak{D}} \mathbf{s}^*(| \xi - \beta\mu |)(I) \\
&\leqq \sum_{\mathfrak{D}} \left\{c/16N + \sum_{\mathfrak{E}_I'} | \xi(J) - \beta(J)\mu(J)|\right\} \\
&\leqq c/16 + \sum_{\mathfrak{D}} \sum_{\mathfrak{E}_I'} | \xi(J) - \gamma(J)\mu(J) | \\
&< c/16 + \int_U | \xi(J) - \gamma(J)\mu(J)| + c/4 \\
&< c/16 + 3c/4 + c/4 = 17c/16.
\end{aligned}$$

Therefore $0 \leqq \int_U s_*(|\xi - \beta\mu|)(I) \leqq \int_U s^*(|\xi - \beta\mu|)(I) = 0$, so that $\int_U |\xi(I) - \beta(I)\mu(I)|$ exists and is 0, which we easily see implies that if V is in $\boldsymbol{F}$, then $\int_V \beta(I)\mu(I)$ exists and is $\xi(V)$.

4. **The characterization theorem.** Definition [1]: If ω is in $\mathfrak{p}^+$, then the statement that ω is μ-refinement-unbounded means that if k is a positive number, then there is a subdivision $\mathfrak{D}$ of U such that if I is in a refinement of $\mathfrak{D}$ and $\mu(I) \neq 0$, then $\omega(I) > k$.

We state a previous theorem of the author [2].

THEOREM 4.A.1. *Suppose σ is in $\mathfrak{p}^+$ and that if each of c and k is a positive number, then there is a subdivision $\mathfrak{D}$ of U such that if $\mathfrak{E}$ is a refinement of $\mathfrak{D}$, then $\sum_{\mathfrak{E}^*} \mu(I) < c$, where $\mathfrak{E}^* = \{I \mid I \text{ in } \mathfrak{E},\ \sigma(I) \leqq k\}$. Then there is a μ-refinement-unbounded element of $\mathfrak{p}^+$.*

We now prove Theorem 4.1, as stated in the introduction.

PROOF OF THEOREM 4.1. We first show that (1) implies (2).

Suppose (1) is true and $\mathfrak{M}$ is a bounded number set and $\boldsymbol{\theta}$ is a function from $\boldsymbol{F}$ into $\overline{\mathfrak{M}}$ and $\int_U \boldsymbol{\theta}(I)\mu(I)$ exists. Let ξ be the element of $\mathfrak{p}$ defined by $\xi(V) = \int_V \boldsymbol{\theta}(I)\mu(I)$. Obviously ξ is in $\mathfrak{p}_A$.

We now show that ξ is μ-dense in $\mathfrak{M}$.

Suppose $0 < c$ and V is in $\boldsymbol{F}$. There is a subdivision $\mathfrak{D}$ of V such that if $\mathfrak{E}$ is a refinement of $\mathfrak{D}$, then $\sum_{\mathfrak{E}} |\xi(I) - \boldsymbol{\theta}(I)\mu(I)| < c/2$. For each I in $\mathfrak{D}$, there is a number $\lambda(I)$ in $\mathfrak{M}$ such that $|\lambda(I) - \boldsymbol{\theta}(I)| < c/[2(\mu(U)+1)]$. This implies that

$$\sum_{\mathfrak{D}} |\xi(I) - \lambda(I)\mu(I)| \leqq \sum_{\mathfrak{D}} |\xi(I) - \boldsymbol{\theta}(I)\mu(I)| + \sum_{\mathfrak{D}} |\boldsymbol{\theta}(I) - \lambda(I)|\mu(I)$$
$$< c/2 + \{c/[2(\mu(U)+1)]\}\mu(U) \leqq c.$$

Therefore ξ is μ-dense in $\mathfrak{M}$ and therefore there is a function ϕ from $\boldsymbol{F}$ into $\mathfrak{M}$ such that if V is in $\boldsymbol{F}$, then $\int_V \phi(I)\mu(I)$ exists and is $\int_V \phi(I)\mu(I)$.

Therefore (1) implies (2).

It is an immediate consequence of Theorem 3.1 that (2) implies (1).

We now show that (2) implies (3).

Suppose (2) is true. Let $\mathfrak{M} = \{z \mid z = 1/q,\ q \text{ a positive integer}\}$. For each V in $\boldsymbol{F}$, let $\boldsymbol{\theta}(V) = 0$. Obviously $\int_V \boldsymbol{\theta}(I)\mu(I) = 0$ for all V in $\boldsymbol{F}$. Since 0 is in $\overline{\mathfrak{M}}$, it follows that there is a function ϕ from $\boldsymbol{F}$ into $\mathfrak{M}$ such that if V is in $\boldsymbol{F}$, then $\int_V \phi(I)\mu(I)$ exists and is $\int_V \boldsymbol{\theta}(I)\mu(I)$.

Now suppose that each of c and k is a positive number. There is a subdivision $\mathfrak{D}$ of U such that if $\mathfrak{E}$ is a refinement of $\mathfrak{D}$, then $\sum_{\mathfrak{E}} \phi(I)\mu(I) < c/k$, so that $\sum_{\mathfrak{E}^*} \mu(I) < c$, where $\mathfrak{E}^* = \{I \mid I \text{ in } \mathfrak{E},\ \phi(I) \geqq 1/k\} = \{I \mid I \text{ in } \mathfrak{E},\ 1/\phi(I) \leqq k\}$.

It follows that the function $1/\phi$ satisfies the hypothesis of Theorem 4.A.1, so that there is a μ-refinement-unbounded element of $\mathfrak{p}^+$. Therefore (2) implies (3).

We now show that (3) implies (2). Suppose (3) is true, i.e. that there is a μ-refinement-unbounded element ω of $\mathfrak{p}^+$, and $\mathfrak{M}$ is a bounded number set and θ is a function from F into $\overline{\mathfrak{M}}$ such that $\int_U \theta(I)\mu(I)$ exists.

There is a subdivision $\mathfrak{D}^*$ of U such that if I is in a refinement of $\mathfrak{D}^*$ and $\mu(I) \neq 0$, then $\omega(I) > 0$.

There is a function ϕ from F into $\mathfrak{M}$ such that if I is in a refinement of $\mathfrak{D}^*$ and $\mu(I) \neq 0$, then $|\theta(I) - \phi(I)| < 1/\omega(I)$.

Suppose $0 < c$ and V is in F. There is a subdivision $\mathfrak{D}$ of V such that if $\mathfrak{E}$ is a refinement of $\mathfrak{D}$, then $|\int_V \theta(I)\mu(I) - \sum_{\mathfrak{E}} \theta(I)\mu(I)| < c/2$. There is a refinement $\mathfrak{D}'$ of $\mathfrak{D}^*$ such that if I is in a refinement of $\mathfrak{D}'$ and $\mu(I) \neq 0$, then $\omega(I) > 2(\mu(U)+1)/c$. There is a subdivision $\mathfrak{D}''$ of V which is a refinement of $\mathfrak{D}$ and a subset of a refinement of $\mathfrak{D}'$. If $\mathfrak{E}$ is a refinement of $\mathfrak{D}''$, then

$$\left| \int_V \theta(I)\mu(I) - \sum_{\mathfrak{E}} \phi(I)\mu(I) \right| \leqq \left| \int_V \theta(I)\mu(I) - \sum_{\mathfrak{E}} \theta(I)\mu(I) \right|$$

$$+ \sum_{\mathfrak{E}} |\, \theta(I) - \phi(I) \,| \mu(I) < c/2 + \sum_{\mathfrak{E}^*} [\mu(I)/\omega(I)] \leqq c/2$$

$$+ \{c/[2(\mu(U) + 1)]\}\mu(U) \leqq c,$$

where $\mathfrak{E}^* = \{I | I \text{ in } \mathfrak{E}, \mu(I) \neq 0\}$.

Therefore $\int_V \phi(I)\mu(I)$ exists and is $\int_V \theta(I)\mu(I)$. Therefore (3) implies (2). Therefore (1), (2) and (3) are equivalent.

References

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