A TOPOLOGICAL APPROACH TO EXTREME POINTS IN FUNCTION SPACES

JOHN CANTWELL[1]

1. **Introduction.** Let X be a Hausdorff space. Denote by $C^n(X)$ the space of all R^n-valued, bounded, continuous functions on X, with the usual supremum norm. If X is compact it is known that the closed unit ball in $C^1(X)$ is the closed convex hull of its extreme points if and only if X is totally disconnected (see Goodner [2]). If X is compact, Phelps [6] has proved that the closed unit ball in $C^2(X)$ is the closed convex hull of its extreme points. If X is compact metric Peck [5] has proved that the closed unit ball in $C^2(X)$ is the convex hull of its extreme points if and only if $\dim X \leqq 1$.

The proofs given by Phelps [6] and Peck [5] rely on measure theory. We remove the measure-theoretic aspects of their arguments and obtain theorems valid for a broader class of spaces X, and, in the case of the result of Peck, for arbitrary n. We prove:

THEOREM I. *Let X be a Hausdorff space. The closed unit ball of $C^2(X)$ is the closed convex hull of its extreme points.*

THEOREM II. *Let X be a normal Hausdorff space. For $n=2, 3, \cdots$ the following two statements are equivalent*:

(i) $\dim X \leqq n-1$,

(ii) *the closed unit ball in $C^n(X)$ is the convex hull of its extreme points.*

The proof of Theorem I is elementary. Theorem II depends on some deep results of dimension theory. We remark that Theorem I is true with $C^2(X)$ replaced by $C^n(X)$ for any $n \geqq 2$. One can see this by using the method of the proof of Lemma 2 to prove a modified Lemma 1. If one replaces "2" by "n" in Lemma 1, one no longer has a continuous unit vector field on S^{n-1} to work with if n is odd. One must therefore use the method of Lemma 2 to obtain f as a convex combination of four functions, instead of two functions, with the desired properties. The proof of Theorem I (with "2" replaced by "n," $n \geqq 2$) then follows as before from this modified Lemma 1.

2. **Notation and preliminaries.** We let D^n (respectively S^{n-1}) denote the closed unit disc (respectively the unit sphere) in R^n. Let $N_\epsilon(x)$

Received by the editors April 4, 1967.

[1] Supported in part by NSF Grant GP-5337.

$=\{y \in R^n \mid |y-x|<\epsilon\}$. All functions considered in this paper are continuous. The closed unit ball in $C^n(X)$, denoted by $U^n(X)$, consists of those $f \in C^n(X)$ such that $f(X) \subset D^n$. It is easy to see that the set of extreme points of $U^n(X)$, denoted by $E^n(X)$, consists of those $f \in C^n(X)$ such that $f(X) \subset S^{n-1}$. By $\dim X$ we mean the Lebesgue covering dimension.

We make use of the following theorems from dimension theory.

THEOREM A. *Let X be a normal space.* $\dim X \leqq n$ *if and only if for each closed subset C of X and each map f from C into S^n, f has an extension f^* from X into S^n.*

THEOREM B. *If X is a normal space,* $\dim X \leqq n-1$, $1 \geqq \delta > 0$, *and f maps X into D^n, then there exists a map g from X into D^n such that*

$$g(x) = f(x) \text{ if } |f(x)| \geqq \delta, \quad |g(x)| = \delta \text{ if } |f(x)| \leqq \delta.$$

The reader is referred to Dowker [1] or Hemmingsen [3] for Theorem A. The reader is also referred to Nagata [4, Theorem III.2, p. 53], for Theorem A in the special case X is metric.

Theorem B follows rather easily from Theorem A. In fact, if f maps X into D^n, define g as follows. Let S_δ^{n-1} (respectively D_δ^n) denote the sphere (respectively closed ball) of radius δ about the origin in R^n. Then $f^{-1}(D_\delta^n)$ is closed in X and thus $\dim f^{-1}(D_\delta^n) \leqq n-1$. Also, $f^{-1}(S_\delta^{n-1})$ is closed in $f^{-1}(D_\delta^n)$. Thus by Theorem A, $f|f^{-1}(S_\delta^{n-1})$ can be extended to a map f^* from $f^{-1}(D_\delta^n)$ to S_δ^{n-1}. Let

$$g(x) = f(x) \text{ if } |f(x)| \geqq \delta, \quad g(x) = f^*(x) \text{ if } |f(x)| \leqq \delta.$$

3. **Proof of Theorem I.** We need an easy lemma.

LEMMA 1. *If f maps X to D^2 and $y \notin f(X)$ for some $y \in D^2$ with $0 \leqq |y| < 1$, then there exist g and h mapping X to $D^2 - N_{1-|y|}(y)$ such that $f = \frac{1}{2}(g+h)$.*

PROOF. Let $v(x)$ be a continuous unit vector field on S^1. If $f(x) \notin N_{1-|y|}(y)$ set $g(x) = h(x) = f(x)$. If $f(x) \in N_{1-|y|}(y)$ set

$$g(x) = f(x) + \lambda(x) v(f(x) - y/|f(x) - y|),$$
$$h(x) = f(x) - \lambda(x) v(f(x) - y/|f(x) - y|),$$

where $\lambda(x) = ((1-|y|)^2 - |f(x)-y|^2)^{1/2}$. Clearly $f = \frac{1}{2}(g+h)$. A calculation shows g and h map X into $D^2 - N_{1-|y|}(y)$.

To prove Theorem I let f map X to D^2 and let $\epsilon > 0$ be given. Let f^* be a map from X to $D^2 - N_\epsilon(1, 0)$ such that $d(f, f^*) \leqq \epsilon$. Let $y_0, \cdots, y_m \in D^2$ be a sequence of points such that $y_0 = (1, 0)$, y_m

$=(0,0)$, $0<d(y_1,y_0)<\epsilon$, and $d(y_{i+1},y_i)<1-|y_i|$, $i=1,\cdots,m-1$. Now apply Lemma 1, m times. First $y_1\notin f^*(X)$. Therefore $f^*=\frac{1}{2}(g+h)$ where $y_2\notin g(X)\cup h(X)$. We then apply Lemma 1 to g and h to get $g=\frac{1}{2}(g^*+h^*)$, $h=\frac{1}{2}(g^{**}+h^{**})$ with $y_3\notin g^*(X)\cup h^*(X)\cup g^{**}(X)\cup h^{**}(X)$. The mth application gives us 2^m functions g_i from X to S^1 such that $f^*=1/2^m\sum g_i$. Theorem I is proved.

4. **Proof of Theorem II.** We first prove a lemma.

LEMMA 2. *If $f\in U^n(X)$, $n=2,3,\cdots$, and $0\notin f(X)$ then there exist $g, h, g^*, h^*\in E^n(X)$ such that $f=(1/4)(g+h+g^*+h^*)$.*

PROOF. We remark that if $(n-1)$ is odd there is a continuous unit tangent vector field on S^{n-1} and the method of Lemma 1 works. In that case we get $f=\frac{1}{2}(g+h)$ with $g, h\in E^n(X)$. For arbitrary n we proceed as follows. Let $\sigma=(0,\cdots,0,-1)$ be the south pole of S^{n-1}. Let $D^n_+=\{(y_1,\cdots,y_n)\in D^n \mid y_n\geqq 0\}$. Let $v=(v_1,\cdots,v_n)$ be a continuous field of unit tangent vectors on $S^{n-1}-\{\sigma\}$. Let λ be a continuous map from $D^n-\{0\}$ to $[0,1]$ such that:

$$\lambda(y)=1 \text{ if } -y_n\leqq |y|(1-|y|^2)^{1/2},\quad \lambda(y)=0 \text{ if } -y_n=|y|.$$

Define

$$\begin{aligned}
g'(x) &= f(x)+\lambda(f(x))(1-|f(x)|^2)^{1/2}v(f(x)/|f(x)|) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{if } -f_n(x)\neq |f(x)|. \\
g'(x) &= f(x) \qquad \text{if } -f_n(x)=|f(x)|. \\
h'(x) &= f(x)-\lambda(f(x))(1-|f(x)|^2)^{1/2}v(f(x)/|f(x)|) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{if } -f_n(x)\neq |f(x)|. \\
h'(x) &= f(x) \qquad \text{if } -f_n(x)=|f(x)|.
\end{aligned}$$

Clearly g' and h' are continuous and $f=\frac{1}{2}(g'+h')$.
Further

$$\begin{aligned}
|g'(x)| &= |f(x)+\lambda(f(x))(1-|f(x)|^2)^{1/2}v(f(x)/|f(x)|)| \\
&= (|f(x)|^2+\lambda(f(x))^2(1-|f(x)|^2))^{1/2}\leqq 1.
\end{aligned}$$

This follows since $f(x)$ and $v(f(x)/|f(x)|)$ are orthogonal and $v(y)$ is a unit vector. Thus $g'(X)\subset D^n$. Similarly, $h'(X)\subset D^n$.

If $-f_n(x)\leqq |f(x)|(1-|f(x)|^2)^{1/2}$ then $\lambda(f(x))=1$ so the above argument gives $|g'(x)|=1$. If $-f_n(x)>|f(x)|(1-|f(x)|^2)^{1/2}$ then $v_n(f(x)/|f(x)|)\leqq f(x)$. In fact if $v_n(f(x)/|f(x)|)>|f(x)|$ under the assumed condition on $f_n(x)$, obtain a contradiction by using the fact that the inner product of $f(x)$ with $v(f(x)/|f(x)|)$ is zero and by estimating the first $n-1$ terms and the last term of this inner product separately, using Cauchy's inequality for the first $n-1$ terms. Therefore $g_n'(x) = f_n(x) + \lambda(f(x))(1-|f(x)|^2)^{1/2} v_n(f(x)/|f(x)|) \leqq f_n(x) + (1-|f(x)|^2)^{1/2}|f(x)| < f_n(x) - f_n(x) = 0$. Therefore in either case $g'(X) \cap D_+^n \subset S^{n-1}$. Similarly, $h'(X) \cap D_+^n \subset S^{n-1}$.

To prove Lemma 2 one now applies the above process to the functions $-g'$ and $-h'$ to get $g' = \frac{1}{2}(g+h)$, $h' = \frac{1}{2}(g^* + h^*)$ with $g(X) \cup h(X) \cup g^*(X) \cup h^*(X) \subset S^{n-1}$.

To prove Theorem II, suppose $\dim X \leqq n-1$. Theorem B implies there is a continuous function g^* from X to D^n such that:

$$\begin{aligned} g^*(x) &= f(x) \qquad \text{if } |f(x)| \geqq 1/3, \\ |g^*(x)| &= 1/3 \qquad \text{if } |f(x)| \leqq 1/3. \end{aligned}$$

Put $h^* = 2f - g^*$. Clearly g^*, $h^* \in U^n(X)$. As above, there exists a continuous function h from X to D^n such that:

$$\begin{aligned} h(x) &= h^*(x) \qquad \text{if } |h^*(x)| \geqq 1/9, \\ |h(x)| &= 1/9 \qquad \text{if } |h^*(x)| < 1/9. \end{aligned}$$

Put $g = 2f - h$. Clearly g, $h \in U^n(X)$, $f = \frac{1}{2}(g+h)$, and $0 \notin h(X)$. Further, since:

$$|g(x) - g^*(x)| = |h(x) - h^*(x)| \leqq 2/9$$

and $|g^*(x)| \geqq 1/3$ for all $x \in X$ it follows that $0 \notin g(X)$. We now apply Lemma 2 to g and h to obtain f as a convex combination of eight elements of $E^n(X)$.

Conversely, suppose $U^n(X)$ is the convex hull of $E^n(X)$. We want to show $\dim X \leqq n-1$. By Theorem A, it suffices to show that for any closed subset C of X and any map f from C to S^{n-1}, f can be extended to a map g from X to S^{n-1}. Suppose f and C are given as above. By Tietze's extension theorem f has an extension f^* from X to D^n, i.e. $f^* \in U^n(X)$. By assumption there exist $f_1, \cdots, f_k \in E^n(X)$ and $r_1, \cdots, r_k \in R$ such that $f^* = \sum r_i f_i$, $\sum r_i = 1$, and $r_1, \cdots, r_k \geqq 0$. For $i = 1, \cdots, k$, $f_i \in E^n(X)$ so f_i maps X to S^{n-1}. Further, if for some $x \in C$, and some $j \in \{1, \cdots, k\}$, $f(x) \neq f_j(x)$ then by the triangle inequality $1 = |f(x)| = |\sum r_i f_i(x)| < \sum |r_i f_i(x)| = \sum r_i = 1$. Thus, for each $j \in \{1, \cdots, k\}$, and each $x \in C$, $f(x) = f_j(x)$. Thus each f_j, $j = 1, \cdots, k$, gives the desired extension of f and Theorem II is proved.

5. It is interesting to note that we have actually proved that if $\dim X \leqq n-1$ any element of $U^n(X)$ is the convex combination of eight elements of $E^n(X)$. If $(n-1)$ is odd the number eight can be

replaced by four. It seems that one should be able to improve on these numbers.

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University of Iowa