AN INEQUALITY BETWEEN THE PERMANENT AND THE DETERMINANT

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A number of relationships between the permanent and other matrix invariants have been discovered [3]. In this paper we prove an inequality between the permanent and the determinant of \( I - A \), where \( I \) is the \( n \)-square identity matrix and \( A \) is an \( n \)-square substochastic matrix.

Suppose that \( A = [a_{ij}] \) is an \( n \)-square matrix. The permanent of \( A \) is defined by

\[
\text{per } A = \sum a_{i_1i_2 \cdots i_n},
\]

where the summation is over all permutations \( i_1 \cdots i_n \) of 1 \cdots n. If \( a_{ij} \geq 0 \) and each row sum of \( A \) is no greater than 1, then \( A \) is a substochastic matrix. If \( A \) is substochastic with each row sum equal to 1, then \( A \) is a stochastic matrix.

If \( r \) is an integer, \( 1 \leq r < n \), let \( Q_{r,n} \) denote the set of all sequences \( \omega = (\omega_1, \omega_2, \ldots, \omega_r) \) of integers for which \( 1 \leq \omega_1 < \omega_2 < \cdots < \omega_r \leq n \). If \( A \) is an \( n \)-square matrix and \( \omega \in Q_{r,n} \) then \( A_\omega \) is the \( (n-r) \)-square submatrix of \( A \) that remains after rows and columns \( \omega_1, \ldots, \omega_r \) are removed.

The following theorem has been proved by the author in these Proceedings [2] and by Brualdi and Newman [1].

**Theorem 1.** If \( A \) is a substochastic matrix, then \( \text{per} (I - A) \geq 0 \).

We use Theorem 1 and mathematical induction to prove the following.

**Theorem 2.** If \( A \) is an \( n \)-square substochastic matrix, then \( \text{per}(I - A) \geq \text{det}(I - A) \geq 0 \).

**Proof.** Clearly, the theorem is true for \( n = 1 \). Let \( A \) be an \( m \)-square substochastic matrix and assume that Theorem 2 is true for all \( n, 1 \leq n < m \). Let \( r_i \) be equal to the \( i \)th row sum of \( A \), \( i = 1, \ldots, m \). Define the \( m \)-square matrix \( D = [d_{ij}] \) by

\[
d_{ij} = d_j = 1 - r_j \quad \text{if} \quad i = j,
\]

\[
= 0 \quad \text{if} \quad i \neq j.
\]

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Obviously $D$ is a nonnegative diagonal matrix and $B$ is a stochastic matrix, where $B = D + A$. We have

$$I - A = D + (I - B).$$

It is well known that

$$\det[D + (I - B)] = d_1d_2 \cdots d_m + \det(I - B) + \sum_{r=1}^{m-1} \sum_{\omega \in Q_{r,m}} d_{\omega_1}d_{\omega_2} \cdots d_{\omega_r} \det(I - B)_\omega.$$

It is easy to prove a similar expansion for the permanent,

$$\per[D + (I - B)] = d_1d_2 \cdots d_m + \per(I - B) + \sum_{r=1}^{m-1} \sum_{\omega \in Q_{r,m}} d_{\omega_1}d_{\omega_2} \cdots d_{\omega_r} \per(I - B)_\omega.$$

Since each row sum of $I - B$ is zero, the columns of $I - B$ are linearly dependent and

$$\det(I - B) = 0.$$

According to Theorem 1,

$$\per(I - B) \geq 0.$$

Since each square submatrix of a stochastic matrix is substochastic, by the inductive assumption,

$$\per(I - B)_\omega \geq \det(I - B)_\omega \geq 0$$

for every $\omega \in Q_{r,m}, r = 1, \cdots, m - 1$. Hence

$$\per(I - A) \geq \det(I - A) \geq 0.$$

References


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