AN INEQUALITY BETWEEN THE PERMANENT AND THE DETERMINANT

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A number of relationships between the permanent and other matrix invariants have been discovered [3]. In this paper we prove an inequality between the permanent and the determinant of $I-A$, where I is the n-square identity matrix and A is an n-square substochastic matrix.

Suppose that $A=[a_{ij}]$ is an n-square matrix. The *permanent* of A is defined by

$$\operatorname{per} A = \sum a_{1i_1}a_{2i_2}\cdots a_{ni_n},$$

where the summation is over all permutations $i_1 \cdots i_n$ of $1 \cdots n$. If $a_{ij}\geqq 0$ and each row sum of A is no greater than 1, then A is a *substochastic* matrix. If A is substochastic with each row sum equal to 1, then A is a *stochastic* matrix.

If r is an integer, $1\leqq r<n$, let $Q_{r,n}$ denote the set of all sequences $\omega=(\omega_1, \omega_2, \cdots, \omega_r)$ of integers for which $1\leqq\omega_1<\omega_2<\cdots<\omega_r\leqq n$. If A is an n-square matrix and $\omega\in Q_{r,n}$ then A_ω is the $(n-r)$-square submatrix of A that remains after rows and columns $\omega_1, \cdots, \omega_r$ are removed.

The following theorem has been proved by the author in these Proceedings [2] and by Brualdi and Newman [1].

THEOREM 1. *If A is a substochastic matrix, then* $\operatorname{per}(I-A)\geqq 0$.

We use Theorem 1 and mathematical induction to prove the following.

THEOREM 2. *If A is an n-square substochastic matrix, then* $\operatorname{per}(I-A)\geqq\det(I-A)\geqq 0$.

PROOF. Clearly, the theorem is true for $n=1$. Let A be an m-square substochastic matrix and assume that Theorem 2 is true for all n, $1\leqq n<m$. Let r_i be equal to the ith row sum of A, $i=1, \cdots, m$. Define the m-square matrix $D=[d_{ij}]$ by

$$\begin{aligned} d_{ij} = d_j &= 1 - r_j \quad \text{if } i = j, \\ &= 0 \quad \text{if } i \neq j. \end{aligned}$$

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Obviously D is a nonnegative diagonal matrix and B is a stochastic matrix, where $B=D+A$. We have

$$I - A = D + (I - B).$$

It is well known that

$$\det[D + (I - B)] = d_1 d_2 \cdots d_m + \det(I - B) + \sum_{r=1}^{m-1} \sum_{\omega \in Q_{r,m}} d_{\omega_1} d_{\omega_2} \cdots d_{\omega_r} \det(I - B)_\omega.$$

It is easy to prove a similar expansion for the permanent,

$$\operatorname{per}[D + (I - B)] = d_1 d_2 \cdots d_m + \operatorname{per}(I - B) + \sum_{r=1}^{m-1} \sum_{\omega \in Q_{r,m}} d_{\omega_1} d_{\omega_2} \cdots d_{\omega_r} \operatorname{per}(I - B)_\omega.$$

Since each row sum of $I-B$ is zero, the columns of $I-B$ are linearly dependent and

$$\det(I - B) = 0.$$

According to Theorem 1,

$$\operatorname{per}(I - B) \geqq 0.$$

Since each square submatrix of a stochastic matrix is substochastic, by the inductive assumption,

$$\operatorname{per}(I - B)_\omega \geqq \det(I - B)_\omega \geqq 0$$

for every $\omega \in Q_{r,m}$, $r=1, \cdots, m-1$. Hence

$$\operatorname{per}(I - A) \geqq \det(I - A) \geqq 0.$$

References

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2. P. M. Gibson, *A short proof of an inequality for the permanent function*, Proc. Amer. Math. Soc. **17** (1966), 535–536.

3. M. Marcus and H. Minc, *Permanents*, Amer. Math. Monthly **72** (1965), 577–591.

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