INDICES OF MAXIMAL SUBGROUPS OF INFINITE SYMMETRIC GROUPS

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Let $M$ be an infinite set with cardinal number $\|M\| = X$, $S(X, Y)$ = \{ $s$: $s$ is a permutation of $M$ with $\| \text{sp} \ s \| < Y$ \}, where $\text{sp} \ s$ = \{ $m \in M$: $s(m) \neq m$ \}. If $X$ is a cardinal number, denote its successor by $X^+$; the smallest infinite cardinal shall be denoted by $d$. Higman [3] has shown that $S(X, d)$ has only one proper subgroup of index less than $X$, the alternating subgroup $A(X)$, and $A(X)$ has no proper subgroups of index less than $X$. Gaughan [2] has extended these results showing that if $d < Y \leq X^+$, then $S(X, Y)$ has no proper subgroups of index less than $X$.

One might conjecture that any proper maximal subgroup of $S(X, Y)$ would have minimal index, namely $X$. It is the purpose of this paper to demonstrate that such is not the case by constructing examples with index greater than $X$. It will also be shown that all intransitive proper maximal subgroups of $S(X, Y)$ do have index $X$.

Hereafter, maximal shall mean proper maximal. Let $M$ be partitioned into $P$ and $Q$ with $\|M\| = X \geq d$, $\|P\| = X$, $\|Q\| = Z$, $0 < Z \leq X$, $d \leq Y \leq X^+$. Let $J(Z) = S(Q) \cdot S(P)$, $J(Y, Z) = J(Z) \cap S(X, Y)$. It has been shown [1] that if $Z < d$, then $J(Y, Z)$ is a maximal subgroup of $S(X, Y)$. In fact every intransitive maximal subgroup of $S(X, Y)$ is of this form. Let $s \in S(X, X^+)$, and define $P_s = \{ x \in P: s(x) \in Q \}$ and $Q_s = \{ y \in Q: s(y) \in P \}$. For each $s \in S(X, X^+)$ define the transfer index of $s = T(s) = \max \{ \|P_s\|, \|Q_s\| \}$. Now define $L(Y, Z)$ as follows: If $d \leq Z < Y \leq X^+, Z < X$, then $L(Y, Z) = \{ s \in S(X, Y): T(s) < Z \}$. If $d < Y \leq Z \leq X$ and $Y$ has an immediate predecessor $Y^-$, then $L(Y, Z) = \{ s \in S(X, Y): T(s) < Y^- \}$. It has been shown [1] that $L(Y, Z)$ is a (transitive) maximal subgroup of $S(X, Y)$.

**Theorem 1.** Let $H$ be an intransitive maximal subgroup of $S(X, Y)$. Then $[S(X, Y): H] = X$.

**Proof.** By [1], $H = J(Y, Z)$ with $0 < Z < d$. By Gaughan [2], $[S(X, Y): H] \geq X$. Since $H$ is intransitive, $A(X) \sqsubseteq H$, so by maximality, $S(X, Y) = A(X) \cdot H$, and $[S(X, Y): H] \leq |A(X)| = X$.

Let $H$ be any subgroup of a group $G$. By $\text{Cl}(H)$ is meant the set of all conjugates of $H$ in $G$, $\text{Cl}(H) = \{ gHg^{-1}: g \in G \}$. It is well known that $[G: N(H)] = |\text{Cl}(H)|$, where $N(H)$ is the normalizer of $H$ in $G$.

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A suitable choice of cardinals $X$, $Y$, and $Z$ will now be made to construct $H = L(Y, Z)$ with $[S(X, Y) : H] > X$.

**Lemma 1.** Let $d \leq Z \leq X$. Let $C$ be a class of subsets of $M$ of cardinality $Z$ such that if $Q_1, Q_2 \in C$, $Q_1 \neq Q_2$, then $|Q_1 - Q_2| = Z$. Then $|C| \leq X^Z$, and there is such a class $C_0$ such that $|C_0| = X^Z$.

**Proof.** The set of all subsets of $M$ of cardinality $Z$ has cardinality $X^Z$. Thus $|C| \leq X^Z$. Let $M = \bigcup_{x \in M} Q_x$ disjointly such that for all $x \in M$, $|Q_x| = Z$. Also let $M = \bigcup_{a \in A} P_a$ disjointly such that $|A| = Z$ and for all $a \in A$, $|P_a| = X$. Let $F$ be the family of nonvoid subsets of $M$ such that for each $D \in F$ and for each $a \in A$, $|D \cap P_a| = 1$. $F$ has the property that $D_1 \subseteq D_2$ if $D_1 \neq D_2$. Furthermore $|F| = X^Z$, and $|D| = Z$ if $D \in F$. Let $E_D = \bigcup_{x \in D} Q_x$, and $C_0 = \{E_D : D \in F\}$. Now $C_0$ has the desired property, and $E_{D_1} = E_{D_2}$ if and only if $D_1 = D_2$, so $|C_0| = |F| = X^Z$.

**Theorem 2.** Let $M$ be partitioned into $P$ and $Q$ as before with $d < Z < Y \leq X^+$, $Z < X$. Then $[S(X, Y) : L(Y, Z)] = X^Z$.

**Proof.** By [1], $L(Y, Z) = J(Y, Z) \cdot S(X, Z)$. Hence for any $s \in S(X, Y)$, $sL(Y, Z)s^{-1} = s[J(Y, Z) \cdot S(X, Z)]s^{-1} = [sJ(Y, Z)s^{-1}] \cdot S(X, Z) = [S(s(Q), Y) \cdot S(s(P), Y)] \cdot S(X, Z)$. Thus $sL(Y, Z)s^{-1} \neq rL(Y, Z)r^{-1}$ if and only if $|s(Q) - r(Q)| = Z$ and $|r(Q) - s(Q)| = Z$. By Lemma 1, there are $X^Z$ subsets of $M$ of cardinality $Z$ having this property and at most $X^Z$ such subsets. Since $Z < Y$, there are $X^Z$ permutations in $S(X, Y)$ mapping $Q$ onto distinct members of this class of subsets of $M$, hence $|Cl(L(Y, Z))| = X^Z$. By the maximality and nonnormality of $L(Y, Z)$, $N(L(Y, Z)) = L(Y, Z)$. Thus $[S(X, Y) : L(Y, Z)] = |Cl(L(Y, Z))| = X^Z$.

**Lemma 2.** There are infinite cardinals $Z$ and $X$ such that $Z < X$ and $X < X^Z$.

**Proof.** Let $Z$ be an infinite cardinal, $A$ a well-ordered set of cardinality $Z$, $\{Z_a\}_{a \in A}$ a family of cardinal numbers such that $Z_a > Z$ for each $a \in A$, and if $a, b \in A$ with $a < b$, then $Z_a < Z_b$. Let $f$ be a one-to-one function from $A$ into $A$ such that $a < f(a)$ for each $a \in A$. Then by the theorem of Koenig and Zermelo, $X = \sum_{a \in A} Z_a < \prod_{a \in A} Z_{f(a)} \leq X^Z$.

**Theorem 3.** There are cardinal numbers $X$ and $Y$, $Y \leq X^+$, such that $S(X, Y)$ contains maximal subgroups with index larger than $X$.

**Proof.** By Lemma 2, choose $X$, $Y$, and $Z$ such that $Z < Y \leq X^+$, $Z < X$, and $X < X^Z$. By Theorem 2, $[S(X, Y) : L(Y, Z)] = X^Z > X$. 

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Theorem 3 is somewhat limited by the choice of $X$ and $Z$. However, for $S(X, X^+)$, maximal subgroups of index greater than $X$ can be demonstrated for any choice of $X \geq d$. Let $s \in S(X, X^+)$ and define $P'_s = P - P(s)$ and $Q'_s = Q - Q(s)$. For each $s \in S(X, X^+)$ define the remainder index of $s = R(s) = \max \{|P'_s|, |Q'_s|\}$. Now define $L(X^+, X) = \{s \in S(X, X^+): T(s) < X \text{ or } R(s) < X\}$. It has been shown [1] that $L(X^+, X)$ is a (transitive) maximal subgroup of $S(X, X^+)$.

**Theorem 4.** For $X \geq d$, $[S(X, X^+): L(X^+, X)] > X$.

**Proof.** Partition $P$ into $P_1$ and $P_2$ such that $|P_1| = |P_2| = X$. By Lemma 1 choose a collection $C$ of subsets of $P_1$ such that $|C| = X^X = 2X$, and if $A, B \in C$ with $A \neq B$, then $|A - B| = X$. Similarly choose a collection $D$ of subsets of $Q$ with the same properties. Let $f$ be a one-to-one function from $C$ onto $D$. For each $A \in C$ choose a corresponding $s \in S(X, X^+)$ such that $s(A) = f(A)$, $s(P - A) = P - A$, $s(Q - f(A)) = Q - f(A)$. Note that $T(s) = R(s) = X$, so $s \in L(X^+, X)$. Let $F$ denote the family of functions so defined. Let $s$ and $r \in F$, $s \neq r$, $s$ and $r$ correspond to $A$ and $B \in C$ respectively. Now $P_{r^{-1}s} \supseteq A - B$, so $X \geq |P_{r^{-1}s}| \geq |A - B| = X$. Thus $T(r^{-1}s) = X$. Also $P_{r^{-1}s} \supseteq (P - A) - B = P - (A \cup B) \supseteq P_2$, so $X \geq |P_{r^{-1}s}| \geq |P_2| = X$. Thus $R(r^{-1}s) = X$. Hence $r^{-1}s \in L(X^+, X)$, so $r$ and $s$ determine different cosets of $L(X^+, X)$ in $S(X, X^+)$. Thus $2X = |S(X, X^+)| \geq [S(X, X^+): L(X^+, X)] \geq |F| = |C| = 2X$, so $[S(X, X^+): L(X^+, X)] = 2X > X$.

**Bibliography**


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