A THEOREM OF NILPOTENT GROUPS

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According to Wiman in [4] and Blackburn in [1], a $p$-group $H$ is said to be of maximal class if the lower central series of $H$,

$$H = H_1 \supset H_2 \supset \cdots \supset H_t \supset H_{t+1} = \{e\}, \quad t \geq 2,$$

has the property that $H/H_2$ is of order $p^2$, and $H_i/H_{i+1}$ is of order $p$, $i = 2, 3, \ldots, t$. Thus, the center of $H$ is $H_t$ whose order is $p$. It seems natural to ask whether one can have a nilpotent group $G$ whose $G_i$, $i \geq 2$, is a $p$-group of maximal class where $G = G_1 \supset G_2 \supset \cdots \supset G_{r+1} = \{e\}$ is the lower central series of $G$. From a theorem of Burnside, [2, p. 241], we know that $G$ cannot be imbedded as the commutator subgroup of a $p$-group. Also, from a result of Hobby in [3, Theorem 1], we know that $G$ cannot be the Frattini subgroup of a $p$-group. In fact, Remark 1 in [3] gives a more general result. Here we shall give a negative answer to our question by proving the following more general theorem:

Let $G$ be a nilpotent group. A subgroup $K$ of $G$ is said to be a $\Psi$-group of $G$ if $K$ is invariant in $G$ and $G_2 \supseteq K \supseteq G_{r-1}$ where $G = G_1 \supset G_2 \supset \cdots \supset G_{r-1} \supset G_r \supset G_{r+1} = \{e\}$ is the lower central series of $G$. Clearly, each $G_i$, $2 \leq i \leq r-1$, is a $\Psi$-group of $G$.

**Theorem.** A nonabelian group whose center is either a cyclic group of order $p$ (a prime) or a cyclic group of infinite order cannot be a $\Psi$-group of a nilpotent group $G$.

**Proof.** Let $G = G_1 \supset G_2 \supset \cdots \supset G_{r-1} \supset G_r \supset G_{r+1} = \{e\}$ be the lower central series of $G$, $H$ be a nonabelian group and $Z(H)$ be the center of $H$ such that $Z(H)$ is either a cyclic group of order $p$ or a cyclic group of infinite order. Suppose the contrary, i.e. $H$ is an invariant subgroup of $G$ and $G_2 \supseteq H \supseteq G_{r-1}$. Since $Z(H)$ is a characteristic subgroup of $H$ and $H$ is invariant in $G$, $Z(H)$ is an invariant subgroup of $G$. Hence, $gZ(H)g^{-1} \subseteq Z(H)$ for every $g \in G$. Let $x$ be a generator of $Z(H)$. Then $gxg^{-1} = x^{n+1}$. We claim that if $Z(H)$ is cyclic of order $p$ then $n \equiv 0 \mod p$ for every $g \in G$, and that if $Z(H)$ is cyclic of infinite order then $n$ must be zero for every $g \in G$. Suppose there is a $g \in G$ such that $[g, x] = gxg^{-1}x^{-1} = x^n \neq e$. Assume

$$[g, [g, [ \cdots [g, x] \cdots ]] = x^{n-1}, \quad (m - 1 \text{ terms})$$

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959
then
\[ [g, [g, [g, \ldots [g, x] \ldots ]] \]

\[(m \text{ terms})\]
\[= (gx^n g^{-1})(x^n g^{-1})^{-1} = (gxg^{-1})^n (x^n g^{-1})^{-1} \]
\[= (x^{n+1})^n (x^{n-1})^{-1} = x^n. \]

For the case of \(Z(H)\) being a cyclic group of order \(p\), \(n \neq 0 \mod p\) implies \(x^n \neq e\) and \(x^m \neq e\) for any positive integer \(m\). For the case of \(Z(H)\) being a cyclic group of infinite order, \(n \neq 0\) implies \(x^n \neq e\) and \(x^m \neq e\) for any positive integer \(m\). Hence, in either case, the lower central series of \(G\) will never reach identity. That is a contradiction to \(G\) being nilpotent. Hence we have \(gxg^{-1} = x\) for every \(g \in G\), i.e. \(x \in Z(G)\) and \(Z(H) \subseteq Z(G)\).

Since \(G/Z(H)\) is nilpotent, we have its lower central series:
\[(G/Z(H)) = (G_1/Z(H)) \supset (G_2/Z(H)) \supset \cdots \supset (G_{r-1}Z(H))/Z(H)) \supset (G_rZ(H))/Z(H) \supset \{e\}.\]
Since \(H\) is nonabelian and \(H \supset G_{r-1}\), there is a nonidentity \(\bar{y} \in (Z(G/Z(H)) \cap (H/Z(H)))\), i.e. if \((G_rZ(H))/Z(H)) \neq \{e\}, then \(\bar{y} \in (G_rZ(H))/Z(H))\). If \((G_rZ(H))/Z(H)) = \{e\}\) then \(\bar{y} \in (G_{r-1}Z(H))/Z(H))\). Since \([\bar{y}, \bar{u}] = e\) for every \(\bar{u} \in (G/Z(H))\), we have \(yxu^{-1}y^{-1} = x^n y u y^{-1} u^{-1} = x^n (y, u)\) where \(n(y, u)\) is an integer, \(\bar{y} = yZ(H)\) and \(\bar{u} = uZ(H)\).

Let \(v\) be any element of \(G\). Then, using the fact \(x \in Z(G)\), we have
\[[y, [u, v]] = y(uvu^{-1}v^{-1}) y^{-1} (uvu^{-1}v^{-1})^{-1} = yuvu^{-1} (v^{-1}yvy^{-1}) y^{-1} uvu^{-1}u^{-1} \]
\[= yuv (u^{-1}y^{-1}uv) y^{-1} u^{-1} x^{-n}(u, v) = yu(yv^{-1}) y^{-1} u^{-1} x^{-n}(u, u) x^{-n}(y, v) \]
\[= x^n (y, u) x^n (y, v) x^{-n}(u, u) x^{-n}(y, v) = e.\]

When \(u\) and \(v\) go through \(G\), \(y\) commutes with every generator of \(G_2\). Hence, \(y \in Z(G_2)\). In particular, \(y\) commutes with every element in \(H\) since \(H \subseteq G_2\). That means \(y \in Z(H)\). That is a contradiction. Hence, \(H\) cannot be a \(\Psi\)-group of a nilpotent group \(G\).

REFERENCES


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