A THEOREM OF NILPOTENT GROUPS

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According to Wiman in [**4**] and Blackburn in [**1**], a p-group H is said to be of maximal class if the lower central series of H,

$$H = H_1 \supset H_2 \supset \cdots \supset H_t \supset H_{t+1} = \{e\}, \qquad t \geqq 2,$$

has the property that H/H_2 is of order p^2, and H_i/H_{i+1} is of order p, $i=2, 3, \cdots, t$. Thus, the center of H is H_t whose order is p. It seems natural to ask whether one can have a nilpotent group G whose G_i, $i \geqq 2$, is a p-group of maximal class where $G=G_1 \supset G_2 \supset \cdots \supset G_{r+1} = \{e\}$ is the lower central series of G. From a theorem of Burnside, [**2**, p. 241], we know that H cannot be imbedded as the commutator subgroup of a p-group. Also, from a result of Hobby in [**3**, Theorem 1], we know that H cannot be the Frattini subgroup of a p-group. In fact, Remark 1 in [**3**] gives a more general result. Here we shall give a negative answer to our question by proving the following more general theorem:

Let G be a nilpotent group. A subgroup K of G is said to be a Ψ-group of G if K is invariant in G and $G_2 \supseteq K \supseteq G_{r-1}$ where $G=G_1 \supset G_2 \supset \cdots \supset G_{r-1} \supset G_r \supset G_{r+1} = \{e\}$ is the lower central series of G. Clearly, each G_i, $2 \leqq i \leqq r-1$, is a Ψ-group of G.

THEOREM. *A nonabelian group whose center is either a cyclic group of order p (a prime) or a cyclic group of infinite order cannot be a Ψ-group of a nilpotent group G.*

PROOF. Let $G=G_1 \supset G_2 \supset \cdots \supset G_{r-1} \supset G_r \supset G_{r+1} = \{e\}$ be the lower central series of G, H be a nonabelian group and $Z(H)$ be the center of H such that $Z(H)$ is either a cyclic group of order p or a cyclic group of infinite order. Suppose the contrary, i.e. H is an invariant subgroup of G and $G_2 \supseteq H \supseteq G_{r-1}$. Since $Z(H)$ is a characteristic subgroup of H and H is invariant in G, $Z(H)$ is an invariant subgroup of G. Hence, $gZ(H)g^{-1} \subseteq Z(H)$ for every $g \in G$. Let x be a generator of $Z(H)$. Then $gxg^{-1} = x^{n+1}$. We claim that if $Z(H)$ is cyclic of order p then $n \equiv 0 \bmod p$ for every $g \in G$, and that if $Z(H)$ is cyclic of infinite order then n must be zero for every $g \in G$. Suppose there is a $g \in G$ such that $[g, x] = gxg^{-1}x^{-1} = x^n \neq e$. Assume

$$[g, [g, [\cdots [g, x] \cdots] = x^{n^{m-1}},$$
$$(m-1 \text{ terms})$$

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then

$$[g,[g,[g,[\cdots[g,x]\cdots]$$
$$(m \text{ terms})$$
$$= (gx^{n^{m-1}}g^{-1})(x^{n^{m-1}})^{-1} = (gxg^{-1})^{n^{m-1}}(x^{n^{m-1}})^{-1}$$
$$= (x^{n+1})^{n^{m-1}}(x^{n^{m-1}})^{-1} = x^{n^m}.$$

For the case of $Z(H)$ being a cyclic group of order p, $n\not\equiv 0 \bmod p$ implies $x^n\neq e$ and $x^{n^m}\neq e$ for any positive integer m. For the case of $Z(H)$ being a cyclic group of infinite order, $n\neq 0$ implies $x^n\neq e$ and $x^{n^m}\neq e$ either for any positive integer m. Hence, in either case, the lower central series of G will never reach identity. That is a contradiction to G being nilpotent. Hence we have $gxg^{-1}=x$ for every $g\in G$, i.e. $x\in Z(G)$ and $Z(H)\subseteq Z(G)$.

Since $G/Z(H)$ is nilpotent, we have its lower central series: $(G/Z(H))=(G_1/Z(H))\supset(G_2/Z(H))\supset\cdots\supset((G_{r-1}Z(H))/Z(H))\supset((G_rZ(H))/Z(H))\supseteq\{\bar e\}$. Since H is nonabelian and $H\supseteq G_{r-1}$, there is a nonidentity $\bar y\in(Z(G/Z(H))\cap(H/Z(H)))$, i.e. if $(G_rZ(H)/Z(H))\neq\{\bar e\}$, then $\bar y\in(G_rZ(H)/Z(H))$. If $(G_rZ(H)/Z(H))=\{\bar e\}$ then $\bar y\in(G_{r-1}Z(H)/Z(H))$. Since $[\bar y,\ \bar u]=\bar e$ for every $\bar u\in(G/Z(H))$, we have $yuy^{-1}u^{-1}=x^nyuy^{-1}u^{-1}=x^{n(y,u)}$ where $n(y,\ u)$ is an integer, $\bar y=yZ(H)$ and $\bar u=uZ(H)$.

Let v be any element of G. Then, using the fact $x\in Z(G)$, we have

$$\begin{aligned}[y,[u,v]] &= y(uvu^{-1}v^{-1})y^{-1}(uvu^{-1}v^{-1})^{-1} = yuvu^{-1}(v^{-1}y^{-1}vy)y^{-1}uv^{-1}u^{-1}\\ &= yuv(u^{-1}y^{-1}uy)y^{-1}v^{-1}u^{-1}x^{-n(y,v)} = yu(vy^{-1}v^{-1}y)y^{-1}u^{-1}x^{-n(y,u)}x^{-n(y,v)}\\ &= x^{n(y,u)}x^{n(y,v)}x^{-n(y,u)}x^{-n(y,v)} = e.\end{aligned}$$

When u and v go through G, y commutes with every generator of G_2. Hence, $y\in Z(G_2)$. In particular, y commutes with every element in H since $H\subseteq G_2$. This means $y\in Z(H)$. That is a contradiction. Hence, H cannot be a Ψ-group of a nilpotent group G.

References

1. N. Blackburn, *On a special class of p-groups*, Acta Math. **100** (1958), 45–92.

2. W. Burnside, *On some properties of groups whose orders are powers of primes*, Proc. London Math. Soc. (2) **11** (1912), 225–245.

3. C. Hobby, *The Frattini subgroup of a p-group*, Pacific J. Math. **10** (1960), 209–212.

4. A. Wiman, *Über p-Gruppen von maximaler Klasse*, Acta Math. **88** (1952), 317–346.

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