A NOTE ON A THEOREM OF GANEA, HILTON AND PETERSON

C. S. HOO

Introduction. Let $X$ be a space. We are interested in the question whether or not the loop space $\Omega X$ and the suspension $\Sigma X$ are homotopy commutative, that is whether or not nil $X \leq 1$, conil $X \geq 1$ respectively. Let $i: X \rightarrow X \vee X$ be the fibre of the inclusion $j: X \vee X \rightarrow X \times X$. Let $\nabla: X \vee X \rightarrow X$ be the folding map. Then in [3], Ganea, Hilton and Peterson proved the following

Theorem 1. Let $X$ be 1-connected. Then nil $X \leq 1$ if and only if $\nabla i = 0$.

Dually, let $q: X \times X \rightarrow X \wedge X$ be the cofibre of the inclusion $j$, and let $\Delta: X \rightarrow X \times X$ be the diagonal map. Let $e': X \wedge X \rightarrow \Omega \Sigma (X \wedge X)$ be the canonical imbedding. Then in [3], the authors also proved

Theorem 2. Let $X$ be 0-connected. Then conil $X \geq 1$ if and only if $e'q \Delta = 0$.

This paper represents an attempt to understand these theorems. Let $c: \Omega(X \vee X) \times \Omega(X \vee X) \rightarrow \Omega(X \vee X)$ be the commutator map. We shall define below a map $\check{c}: \Sigma(\Omega X \times \Omega X) \rightarrow X \vee X$ obtained from $c$. Applying the co-Hopf construction, we have a map $H(\check{c}): \Omega \Sigma (\Omega X \times \Omega X) \rightarrow \Omega (X \vee X)$. Then we prove

Theorem 3. $c = \Omega(\nabla i)H(\check{c})e': \Omega X \times \Omega X \rightarrow \Omega X$, the commutator map.

We observe, of course, that the condition for nil $X \leq 1$ is precisely $c = 0$. Dually, let $c': \Sigma(X \times X) \rightarrow \Sigma(X \times X) \vee \Sigma(X \times X)$ be the cocommutator map. This gives a map $\check{c}: X \times X \rightarrow \Omega(\Sigma X \vee \Sigma X)$. The Hopf construction then gives a map $J(\check{c}'): \Sigma(X \wedge X) \rightarrow \Sigma \Omega (\Sigma X \vee \Sigma X)$. Let $e: \Sigma \Omega (\Sigma X \vee \Sigma X) \rightarrow \Sigma X \vee \Sigma X$ be the map having $1_{\Sigma X \vee \Sigma X}$ as its adjoint. Let us denote the cocommutator product $\Sigma X \rightarrow \Sigma X \vee \Sigma X$ by $c'$ also. The condition for conil $X \geq 1$ is precisely $c' = 0$. We prove

Theorem 4. $c' = eJ(\check{c}')\Sigma(q \Delta): \Sigma X \rightarrow \Sigma X \vee \Sigma X$.

We work in the category of spaces with base point and having the homotopy type of countable CW-complexes. For simplicity, we shall frequently use the same symbol for a map and its homotopy class.

Received by the editors April 18, 1967.

909
1. Let $A, B$ be spaces. We have the fibration $A \triangleright B \xrightarrow{i} A \vee B$. We can find a map $\chi: \Omega(A \times B) \to \Omega(A \vee B)$ such that $(\Omega j)\chi \simeq 1_{\Omega(A \times B)}$. In fact we can take $\chi = \Omega(i_A p_A) + \Omega(i_B p_B)$ where $p_A, p_B$ are the projections of $A \times B$ onto the factors and $i_A: A \to A \vee B, i_B: B \to A \vee B$ are the inclusions. The exact sequence of the fibration now shows that there exists a unique element $[g] \in [\Omega(A \vee B), \Omega(A \triangleright B)]$ such that $1_{\Omega(A \triangleright B)} = (\Omega i)g + \chi(\Omega j)$.

Now for any space $X$ and a map $f: X \to A \vee B$ we can form the map $H(f) = g(\Omega f): \Omega X \to \Omega(A \triangleright B)$. We shall call this the co-Hopf construction. The element $[H(f)]$ is the unique element of $[\Omega X, \Omega(A \triangleright B)]$ satisfying $[\Omega f] = (\Omega i)\#[H(f)] + [\chi] = (\Omega i)\#[H(f)] + [\Omega(i_A \pi_A f)] + [\Omega(i_B \pi_B f)]$ where $\pi_A: A \vee B \to A, \pi_B: A \vee B \to B$ are induced by the projections onto the factors.

For spaces $X, Y$ we have a bijection $\tau: [\Sigma X, Y] \to [X, \Omega Y]$ which takes each map to its adjoint. Suppose $X$ is a given space. We have a projection $p: \Sigma \Omega X \to X$ such that $\tau(p) = 1_{\Omega X}$. Let $p_1 = i_1 p, p_2 = i_2 p$ where $i_1, i_2: X \to X \vee X$ are the injections in the first and second copies of $X$ respectively. Let $\varsigma: \Omega(X \vee X) \times \Omega(X \vee X) \to \Omega(X \vee X)$ be the commutator map. Then we can form the map $\varsigma = \tau^{-1}(\varsigma(\tau(p_1) \times \tau(p_2))): \Sigma(\Omega X \times \Omega X) \to X \vee X$. It is now easily verified that $\nabla \varsigma = \tau^{-1}(\varsigma)$. The co-Hopf construction, applied to $\varsigma$, gives an element $H(\varsigma): \Omega(\varsigma(\Omega X \times \Omega X)) \to \Omega(X \triangleright B)$. Let $e': \Omega X \times \Omega X \to \Omega(\varsigma(\Omega X \times \Omega X))$ be such that $e' = \tau(1_{\Omega X \times \Omega X})$. It is easily seen that $\Omega(\tau^{-1}(\varsigma))e' = c: \Omega X \times \Omega X \to \Omega X$, the commutator map. Since $\nabla \varsigma = \tau^{-1}(\varsigma)$, Theorem 3 follows immediately from

**Theorem 5.** $\Omega(\nabla \varsigma) = \Omega(\nabla i)H(\varsigma): \Omega(\varsigma(\Omega X \times \Omega X)) \to \Omega X$.

**Proof.** $H(\varsigma)$ satisfies $\Omega \varsigma = (\Omega i)H(\varsigma) + \Omega(\pi_1 \tau) + \Omega(\pi_2 \tau)$ where $\pi_1, \pi_2: X \vee X \to X$ are induced by the projections onto the factors, and $i_1, i_2: X \to X \vee X$ are the imbeddings in the first and second copies of $X$ respectively. We have $\Omega(\nabla i) = \Omega(\nabla i)H(\varsigma) + \Omega(\nabla \pi_1 \tau) + \Omega(\nabla \pi_2 \tau)$. Let $\phi$ be the loop multiplication on $\Omega X$ and $\mu$ the loop inverse. Then a simple check shows that $\tau(\nabla \pi_1 \tau) = \phi(1 \times *) \Delta \phi(1 \times *) \Delta \mu \Delta r_1$ where $\Delta$ is the diagonal map and $r_1: \Omega X \times \Omega X \to \Omega X$ is the projection onto the first factor. Since $\phi(1 \times *) \Delta \simeq 1$ and $\phi(1 \times \mu) \Delta \simeq \ast$, we have $\tau(\nabla \pi_1 \tau) = 0$. Hence $\nabla \pi_1 \tau = 0$. Similarly $\nabla \pi_2 \tau = 0$. It follows then that $\Omega(\nabla \varsigma) = \Omega(\nabla i)H(\varsigma)$.

Theorems 1 and 3 are the immediate

**Corollary.** Let $X$ be 1-connected. If $\Omega(\nabla i) = 0$, then $\nabla i = 0$.

**Remark.** In [3], it is shown that there exist maps $a, b$ such that $ba = 1, ib = \varsigma$. It is clear from the above that $H(\varsigma) = \Omega b$. 
2. We now dualise. Let \( p_1, p_2 : X \times X \to X \) be the projections, and let \( e_i = e' \circ p_i \) where \( e' : X \to \Omega \Sigma X \) is the canonical imbedding. Let \( e' \) be the cocommutator map \( \Sigma(X \times X) \to \Sigma(X \times X) \). Let \( e' = \tau \{ (r^{-1}(e_1) \lor r^{-1}(e_2)) e' \} : X \times X \to \Omega \Sigma X \lor \Sigma X \). Then \( e' \Delta = \tau(e') \) where \( \Delta \) is the diagonal map.

Let \( A, B \) be spaces. We consider the cofibration \( A \lor B \to A \times B \). There exists a map \( p : \Sigma(A \times B) \to \Sigma(A \lor B) \) such that \( p(Q^2) \approx 1_{\Sigma(A \lor B)} \). The exact sequence of the cofibration now shows that \( (\Sigma q)^f \) is a monomorphism. Dual to the above, we now see that there exists a unique element \( [d] \in [\Sigma(A \lor B), \Sigma(A \times B)] \) satisfying \( 1_{\Sigma(A \lor B)} = (\Sigma q)^f + (\Sigma j)^f \).

Given a map \( f : A \times B \to X \) we can now define \( J(f) = (\Sigma f) d : \Sigma(A \lor B) \to \Sigma X \). We shall call \( J(f) \) the map obtained from \( f \) by the Hopf construction. The element \( [J(f)] \) is the unique element satisfying \( [\Sigma(f)] = (\Sigma q)^f J(f) + [\Sigma(\Sigma j)^f] + [\Sigma(f p_A^1) + [\Sigma(f p_B^1)] \) where \( p_A, p_B : A \times B \to A \lor B \) are induced by the projections onto the first and second coordinates respectively. We can now consider the element \( J(e') : \Sigma(X \lor X) \to \Sigma \Omega(\Sigma X \lor \Sigma X) \). We have \( J(e') \Sigma(q \Delta), \Sigma(e' \Delta) : \Sigma X \to \Sigma \Omega(\Sigma X \lor \Sigma X) \). Let \( e : \Sigma \Omega(\Sigma X \lor \Sigma X) \to \Sigma X \lor \Sigma X \) be such that \( \tau(e) = 1_{\Sigma X \lor \Sigma X} \). Let \( c' : \Sigma X \to \Sigma X \lor \Sigma X \) be the cocommutator map. It is now easily checked that \( e \Sigma(\tau(e')) = c' \). Since \( e' \Delta = \tau(e') \). Theorem 4 follows immediately from

\textbf{Theorem 6.} \( \Sigma(e' \Delta) = J(e') \Sigma(q \Delta) : \Sigma X \to \Sigma \Omega(\Sigma X \lor \Sigma X) \).

\textbf{Proof.} The proof is completely dual to that of Theorem 5, and we shall omit it.

\textbf{Remark 1.} In [3], it is shown that we can find maps \( a', b' \) such that \( b' a' = 1, a'e' q = e' \). It is easily seen that \( J(e') = \Sigma(a' e') \).

\textbf{Remark 2.} Theorems 3 and 4 give other conditions for nil \( X \leq 1 \), conil \( X \leq 1 \) respectively, namely whenever some combination of factors in the factorizations of \( c, c' \) is null-homotopic.

\textbf{References}


University of Alberta