A NOTE ON A THEOREM OF GANEA, HILTON AND PETERSON

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Introduction. Let $X$ be a space. We are interested in the question whether or not the loop space $\Omega X$ and the suspension $\Sigma X$ are homotopy commutative, that is whether or not nil $X \leq 1$, conil $X \leq 1$ respectively. Let $i: X \ni X \rightarrow X \vee X$ be the fibre of the inclusion $j: X \vee X \rightarrow X \times X$. Let $\nabla: X \vee X \rightarrow X$ be the folding map. Then in [3], Ganea, Hilton and Peterson proved the following

**Theorem 1.** Let $X$ be 1-connected. Then nil $X \leq 1$ if and only if $\nabla i = 0$.

Dually, let $q: X \times X \rightarrow X \wedge X$ be the cofibre of the inclusion $j$, and let $\Delta: X \rightarrow X \times X$ be the diagonal map. Let $e': X \wedge X \rightarrow \Omega \Sigma (X \wedge X)$ be the canonical imbedding. Then in [3], the authors also proved

**Theorem 2.** Let $X$ be o-connected. Then conil $X \leq 1$ if and only if $e' q \Delta = 0$.

This paper represents an attempt to understand these theorems. Let $c: \Omega (X \vee X) \times \Omega (X \vee X) \rightarrow \Omega (X \vee X)$ be the commutator map. We shall define below a map $\tilde{c}: \Sigma (\Omega X \times \Omega X) \rightarrow X \vee X$ obtained from $c$. Applying the co-Hopf construction, we have a map $H(\tilde{c}): \Omega \Sigma (\Omega X \times \Omega X) \rightarrow \Omega (X \ni X)$. Then we prove

**Theorem 3.** $c = \Omega (\nabla i) H(\tilde{c}) e': \Omega X \times \Omega X \rightarrow \Omega X$, the commutator map.

We observe, of course, that the condition for nil $X \leq 1$ is precisely $c = 0$. Dually, let $c': \Sigma (X \times X) \rightarrow \Sigma (X \times X) \vee \Sigma (X \times X)$ be the cocommutator map. This gives a map $\tilde{c}': X \times X \rightarrow \Sigma (\Sigma X \vee \Sigma X)$. The Hopf construction then gives a map $J(\tilde{c}'): \Sigma (X \wedge X) \rightarrow \Sigma \Omega (\Sigma X \vee \Sigma X)$. Let $e: \Sigma \Omega (\Sigma X \vee \Sigma X) \rightarrow \Sigma X \vee \Sigma X$ be the map having $1_{\Omega (\Sigma X \vee \Sigma X)}$ as its adjoint. Let us denote the cocommutator product $\Sigma X \rightarrow \Sigma X \vee \Sigma X$ by $c'$ also. The condition for conil $X \leq 1$ is precisely $c' = 0$. We prove

**Theorem 4.** $c' = e J(\tilde{c}') \Sigma (q \Delta): \Sigma X \rightarrow \Sigma X \vee \Sigma X$.

We work in the category of spaces with base point and having the homotopy type of countable CW-complexes. For simplicity, we shall frequently use the same symbol for a map and its homotopy class.

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1. Let $A, B$ be spaces. We have the fibration $A \triangleright B \rightarrow A \vee B$. We can find a map $\chi: \Omega(A \times B) \rightarrow \Omega(A \vee B)$ such that $(\Omega j)\chi \cong 1_{\Omega(A \vee B)}$. In fact we can take $\chi = \Omega(i_A p_A) + \Omega(i_B p_B)$ where $p_A, p_B$ are the projections of $A \times B$ onto the factors and $i_A: A \rightarrow A \vee B, i_B: B \rightarrow A \vee B$ are the inclusions. The exact sequence of the fibration now shows that there exists a unique element $[g] \in [\Omega(A \vee B), \Omega(A \triangleright B)]$ such that $1_{\Omega(A \vee B)} = (\Omega i)g + \chi(\Omega j)$.

Now for any space $X$ and a map $f: X \rightarrow A \vee B$ we can form the map $H(f) = g(\Omega f): \Omega X \rightarrow \Omega(A \triangleright B)$. We shall call this the co-Hopf construction. The element $[H(f)]$ is the unique element of $[\Omega X, \Omega(A \triangleright B)]$ satisfying $[\Omega f] = (\Omega i)\#[H(f)] + [x\Omega(\Omega f)] = (\Omega i)\#[H(f)] + [\Omega(i_A \pi_A f)] + [\Omega(i_B \pi_B f)]$ where $\pi_A: A \vee B \rightarrow A$, $\pi_B: A \vee B \rightarrow B$ are induced by the projections onto the factors.

For spaces $X, Y$ we have a bijection $\tau: [\Sigma X, Y] \rightarrow [X, \Omega Y]$ which takes each map to its adjoint. Suppose $X$ is a given space. We have a projection $p: \Sigma \Omega X \rightarrow X$ such that $\tau(p) = 1_{\Omega X}$. Let $p_1 = i_1 p$, $p_2 = i_2 p$ where $i_1, i_2: X \rightarrow X \vee X$ are the injections in the first and second copies of $X$ respectively. Let $c: \Omega(X \vee X) \times \Omega(X \vee X) \rightarrow \Omega(X \vee X)$ be the commutator map. Then we can form the map $\bar{c} = \tau^{-1}\{c(\tau(p_1) \times \tau(p_2))\}: \Sigma(\Omega X \times \Omega X) \rightarrow X \vee X$. It is now easily verified that $\nabla c = \tau^{-1}(c)$. The co-Hopf construction, applied to $\bar{c}$, gives an element $H(c): \Omega(\Sigma(\Omega X \times \Omega X) \rightarrow \Omega(X \vee X)$. Let $e': \Omega X \times \Omega X \rightarrow \Omega(\Sigma(\Omega X \times \Omega X) be such that $e' = \tau(1_{\Sigma(\Omega X \times \Omega X)})$. It is easily seen that $\Omega(\tau^{-1}(c))e' = c: \Omega X \times \Omega X \rightarrow \Omega X$, the commutator map. Since $\nabla c = \tau^{-1}(c)$, Theorem 3 follows immediately from

**Theorem 5.** $\Omega(\nabla c) = \Omega(\nabla i)H(c): \Omega(\Sigma(\Omega X \times \Omega X) \rightarrow \Omega X$.

**Proof.** $H(c)$ satisfies $\Omega c = (\Omega i)\Omega H(c) + \Omega(i_1 \pi_1 c) + \Omega(i_2 \pi_2 c)$ where $\pi_1, \pi_2: X \vee X \rightarrow X$ are induced by the projections onto the factors, and $i_1, i_2: X \rightarrow X \vee X$ are the imbeddings in the first and second copies of $X$ respectively. We have $\Omega(\nabla c) = \Omega(\nabla i)H(c) + \Omega(\nabla i_1 \pi_1 c) + \Omega(\nabla i_2 \pi_2 c)$. Let $\phi$ be the loop multiplication on $\Omega X$ and $\mu$ the loop inverse. Then a simple check shows that $\tau(\nabla i_1 \pi_1 c) = \phi(1 \times*) \Delta \times \phi(1 \times*) \Delta \mu$ where $\Delta$ is the diagonal map and $r_1: \Omega X \times \Omega X \rightarrow \Omega X$ is the projection onto the first factor. Since $\phi(1 \times*) \Delta \simeq 1$ and $\phi(1 \times \mu) \Delta \simeq *$, we have $\tau(\nabla i_1 \pi_1 c) = 0$. Hence $\nabla i_1 \pi_1 c = 0$. Similarly $\nabla i_2 \pi_2 c = 0$. It follows then that $\Omega(\nabla c) = \Omega(\nabla i)H(c)$.

Theorems 1 and 3 are the immediate

**Corollary.** Let $X$ be 1-connected. If $\Omega(\nabla i) = 0$, then $\nabla i = 0$.

**Remark.** In [3], it is shown that there exist maps $a, b$ such that $ba = 1, ib = e$. It is clear from the above that $H(e) = \Omega b$. License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
2. We now dualise. Let $p_1, p_2 : X \times X \to X$ be the projections, and let $e_i = e'_i p_i$ where $e' : X \to \Omega \Sigma X$ is the canonical imbedding. Let $c'$ be the cocommutator map $\Sigma(X \times X) \to \Sigma(X \times X) \vee \Sigma(X \times X)$. Let $c' = \tau \{ (r^{-1}(e_1) \vee r^{-1}(e_2))c' \} : X \times X \to \Omega(\Sigma X \vee \Sigma X)$. Then $c' \Delta = \tau(c')$ where $\Delta$ is the diagonal map.

Let $A, B$ be spaces. We consider the cofibration $A \vee B \to A \times B \to A \wedge B$. There exists a map $p : \Sigma(A \times B) \to \Sigma(A \wedge B)$ such that $p(\Sigma j) \sim 1_{\Sigma(A \vee B)}$. The exact sequence of the cofibration now shows that $(\Sigma q)^g$ is a monomorphism. Dual to the above, we now see that there exists a unique element $[d] \in [\Sigma(A \wedge B), \Sigma(A \times B)]$ satisfying $1_{\Sigma(A \times B)} = d(\Sigma q) + (\Sigma j)p$.

Given a map $f : A \times B \to X$ we can now define $J(f) = (\Sigma f) d : \Sigma(A \wedge B) \to \Sigma X$. We shall call $J(f)$ the map obtained from $f$ by the Hopf construction. The element $[J(f)]$ is the unique element satisfying $[\Sigma f] = (\Sigma q)^g[J(f)] + [\Sigma (fj p_A)] + [\Sigma (fj p_B)]$ where $p_A, p_B : A \times B \to A \vee B$ are induced by the projections onto the first and second coordinates respectively. We can now consider the element $J(c') : \Sigma(X \wedge X) \to \Sigma(\Sigma X \vee \Sigma X)$. We have $J(c') \Sigma(q \Delta), \Sigma(c' \Delta) : \Sigma X \to \Sigma(\Sigma X \vee \Sigma X)$. Let $e : \Sigma X \vee \Sigma X \to \Sigma X \vee \Sigma X$ be such that $\tau(e) = 1_{\Sigma X \vee \Sigma X}$. Let $c' : \Sigma X \to \Sigma X \vee \Sigma X$ be the cocommutator map. It is now easily checked that $e\Sigma(\tau(c')) = c'$. Since $c' \Delta = \tau(c')$, Theorem 4 follows immediately from

**Theorem 6.** $\Sigma(c' \Delta) = J(c') \Sigma(q \Delta) : \Sigma X \to \Sigma(\Sigma X \vee \Sigma X)$.

**Proof.** The proof is completely dual to that of Theorem 5, and we shall omit it.

**Remark 1.** In [3], it is shown that we can find maps $a', b'$ such that $b'a' = 1$, $a'e'q = c'$. It is easily seen that $J(c') = \Sigma(a'e')$.

**Remark 2.** Theorems 3 and 4 give other conditions for nil $X \leq 1$, conil $X \leq 1$ respectively, namely whenever some combination of factors in the factorizations of $c, c'$ is null-homotopic.

**References**


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