A NOTE ON FINITE GROUPS IN WHICH NORMALITY IS TRANSITIVE

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1. **Introduction.** We will say that a group G satisfies the *condition* $\mathcal{C}_p$ (where p is a prime) if every subgroup of a Sylow p-subgroup P of G is normal in the normalizer of P. Here we wish to consider the relation between the condition $\mathcal{C}_p$ and the *class* $\mathfrak{T}$ of all groups in which normality is a transitive relation. More precisely $G \in \mathfrak{T}$ if and only if $H \lhd K \lhd G$ always implies that $H \lhd G$. Our object here is to prove

THEOREM 1. *A finite group which satisfies $\mathcal{C}_p$ for all p is a soluble $\mathfrak{T}$-group.*

Let $\bar{\mathfrak{T}}$ denote the class of all groups G for which $H \lhd K \lhd L \leqq G$ always implies that $H \lhd L$: in short $\bar{\mathfrak{T}}$ is the largest subclass of $\mathfrak{T}$ that is closed with respect to forming subgroups. Now every finite soluble $\mathfrak{T}$-group is a $\bar{\mathfrak{T}}$-group [2, Satz 4][1] and it is obvious that a finite $\bar{\mathfrak{T}}$-group satisfies $\mathcal{C}_p$ for all p, since every subgroup of a finite p-group is subnormal. Consequently we have

THEOREM 1*. *If G is a finite group, the following are equivalent statements.*

(i) *G is soluble $\mathfrak{T}$-group.*

(ii) *G is a $\bar{\mathfrak{T}}$-group.*

(iii) *G satisfies $\mathcal{C}_p$ for all p.*

Every soluble $\mathfrak{T}$-group is metabelian [4, Theorem 2.3.1], so Theorem 1* yields the following information about infinite $\bar{\mathfrak{T}}$-groups.

COROLLARY. *A locally finite $\bar{\mathfrak{T}}$-group is soluble.*

The proof of Theorem 1 uses the Schur-Zassenhaus splitting theorem, Burnside's theorem on the existence of a normal complement of a Sylow subgroup that lies in the centre of its normalizer and Grün's First Theorem [3]. In addition we need some simple facts about $\mathfrak{T}$-groups, the first of which has already been mentioned.

(A) Soluble $\mathfrak{T}$-groups are metabelian.

(B) Let $N \lhd G$ where every subnormal subgroup of N is normal in G, G/N belongs to $\mathfrak{T}$ and the order of N is prime to its index. Then G belongs to $\mathfrak{T}$ [4, Lemma 5.2.2].

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[1] However, not every finite $\mathfrak{T}$-group or infinite soluble $\mathfrak{T}$-group is in $\bar{\mathfrak{T}}$, [4].

(C) If A is a finite abelian group and α is a *power automorphism* of A (i.e. an automorphism which leaves every subgroup of A invariant), then there exists a positive integer m such that $a^\alpha = a^m$ for all $a \in A$, [2, p. 88].

2. $\mathcal{C}_p$ **and** p**-nilpotence.** We will prove Theorem 1 via two preliminary results connecting $\mathcal{C}_p$ with the notion of p-nilpotence. (Recall that a finite group G is *p-nilpotent* if it has a normal subgroup of index a power of p and of order prime to p.)

THEOREM 2. *If the finite group G satisfies $\mathcal{C}_p$ where p is the smallest prime dividing the order of G, then G is p-nilpotent.*

PROOF. Let P be a Sylow p-subgroup of G and let $N = N_G(P)$, its normalizer in G. Every subgroup of P is normal in N, so P is either abelian or Hamiltonian. Suppose that p is odd, so that P is abelian. Elements of N induce power automorphisms in P of order prime to p and therefore dividing $p-1$. Since p is the smallest prime dividing the order of G, P lies in the centre of N and by Burnside's theorem P has a normal complement, i.e. G is p-nilpotent. Now let $p=2$. By the Schur-Zassenhaus theorem N splits over P and so we may write $N = PH$, with $P \cap H = 1$. A power automorphism of P has order a power of 2 (whether or not P is abelian) and H has odd order. Hence H centralizes P and $N = P \times H$.

If P is abelian, it is central in N and we can again use Burnside's theorem. Suppose therefore that P is Hamiltonian.

Let σ denote the transfer of G into P/P' and let $K = \operatorname{Ker} \sigma$. Then

$$G/K \cong P/P^*$$

where

$$P^* = (P \cap N') \prod_{g \in G} (P \cap (P')^g)$$

by Grün's First Theorem. $P \cap (P')^g$ and $(P \cap (P')^g)^{g^{-1}}$ are both normal in P and hence by a standard "Sylow" argument [3, Lemma 14.3.1] these subgroups are conjugate in N. By $\mathcal{C}_p$

$$P \cap (P')^g = (P \cap (P')^g)^{g^{-1}}$$

and therefore $P \cap (P')^g \leqq P'$ for all $g \in G$. On the other hand since $N = P \times H$, $P \cap N' = P'$. Thus we conclude that $P^* = P'$, which by the structure of Hamiltonian groups [3, Theorem 12.5.4] has order 2. Since $G/K \cong P/P'$, P' is a Sylow 2-subgroup of K and clearly it must lie in the centre of $N_K(P')$. Therefore P' has a normal complement in K

and this is obviously also a normal complement for P in G. This completes the proof.

The solubility of groups of odd order yields the following

COROLLARY. *A finite group satisfying* $\mathfrak{C}_2$ *is soluble.*

The alternating group of degree 5 satisfies $\mathfrak{C}_3$ and $\mathfrak{C}_5$ but not $\mathfrak{C}_2$, so the hypothesis that p is "smallest" cannot be omitted from the statement of Theorem 2. Let us say that a group is *p-perfect* if it has no nontrivial abelian p-factor groups. A finite group that is both p-nilpotent and p-perfect has order prime to p, so that these properties represent extremes of behaviour for finite groups. It is therefore interesting that the condition $\mathfrak{C}_p$ forces a finite group to one of those extremes. This is

THEOREM 3.[2] *If the finite group G satisfies $\mathfrak{C}_p$, G is either p-nilpotent or p-perfect.*

PROOF. Let P be a Sylow p-subgroup of G and let $N = N_G(P)$. If P lies in the centre of N, G is p-nilpotent, so we can suppose that $[P, x] \neq 1$ for some $x \in N$. In addition we may assume that $p > 2$, otherwise G would again be p-nilpotent, by Theorem 2. Hence P is abelian and by (C) there is an $m > 0$ such that $a^x = a^m$ for any $a \in P$. If $m \equiv 1 \bmod p$, $m^{p^i} \equiv 1 \bmod p^{i+1}$ for $i \geqq 0$, and the automorphism induced in P by x would have order a power of p; since $|N:P|$ is prime to p, this automorphism would have to be trivial. Hence $m \not\equiv 1 \bmod p$ and so

$$\langle [a, x] \rangle = \langle a^{m-1} \rangle = \langle a \rangle$$

for all $a \in P$. Consequently $P \leqq G'$, which shows that G is p-perfect.

PROOF OF THEOREM 1. Suppose that G is a finite group of least order such that G satisfies $\mathfrak{C}_p$ for all p and yet G is *not* a soluble $\mathfrak{J}$-group. Let p be the smallest prime dividing the order of G, so that G is p-nilpotent by Theorem 2. We can write $G = PH$ where $H \lhd G$, $H \cap P = 1$ and P is a Sylow p-subgroup of G. The order of H is prime to its index in G, so H satisfies all the conditions $\mathfrak{C}$ and is a soluble $\mathfrak{J}$-group by minimality of G; thus G is certainly soluble and to obtain a contradiction we have only to show that G belongs to $\mathfrak{J}$. Suppose that H is nilpotent and hence abelian (being of odd order). Let q be a prime dividing the order of H; then Q, the q-primary component of H, is the unique Sylow q-subgroup of G and hence each subgroup of Q is normal in G, by $\mathfrak{C}_q$. G splits over Q with, say, $G = KQ$ and $K \cap Q$

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[2] See [6, p. 59] for a special case of this result.

$=1$; K satisfies all the conditions $\mathfrak{C}$ and is therefore a $\mathfrak{J}$-group, so that G/Q is a $\mathfrak{J}$-group. That G belongs to $\mathfrak{J}$ now follows by (B). Next we suppose that H is non-nilpotent, so there is a prime q dividing the order of H such that H is not q-nilpotent. By (A) H' is abelian and Theorem 3 shows that H is q-perfect and so H/H' has order prime to q. Hence Q, the q-primary component of H', is the unique Sylow q-subgroup of G and has each of its subgroups normal in G. G splits over Q and G/Q belongs to $\mathfrak{J}$ by minimality. Finally G belongs to $\mathfrak{J}$ by (B) as before.

3. **Pronormal subgroups.** A subgroup H of a group G is said to be *pronormal* in G if for any $g \in G$, H and H^g are already conjugate in their join $\langle H, H^g\rangle$. There is a link between pronormality and the condition $\mathfrak{C}_p$ which was pointed out by Dr. J. S. Rose (to whom the author is indebted for several useful comments).

LEMMA (J. S. ROSE). *A finite group G satisfies $\mathfrak{C}_p$ if and only if every p-subgroup is pronormal in G.*

PROOF. Assume that G satisfies $\mathfrak{C}_p$ and let P_0 be any p-subgroup of G. Let $g \in G$; we show that P_0 and P_0^g are conjugate in $J=\langle P_0, P_0^g\rangle$. Let P_1 be a Sylow p-subgroup of J containing P_0. Then for some $x \in J$, $P_0^g \leq P_1^x$; hence P_0 and $P_0^{gx^{-1}}$ are both contained in P_1. Let P be a Sylow p-subgroup of G containing P_1. By $\mathfrak{C}_p$ P_0 and $P^{gx^{-1}}$ are both normal in P and hence are conjugate in $N_G(P)$; by $\mathfrak{C}_p$ again, $P_0 = P_0^{gx^{-1}}$ and so $P_0^g = P_0^x$. To prove the converse note that pronormality and subnormality together imply normality.

COROLLARY. *For finite groups the condition $\mathfrak{C}_p$ is inherited by subgroups and homomorphic images.*

PROOF. The subgroup part is clear. Let H/N be a p-subgroup of G/N where G satisfies $\mathfrak{C}_p$ and let P be a Sylow p-subgroup of H. By comparison of orders $H=PN$. P is pronormal in G and clearly this implies that H/N is pronormal in G/N. It will be noted that in the proof of Theorem 1 the fact that $\mathfrak{C}_p$ passes to subgroups was used only in situations where this was obvious.[3] Observe also that the main theorem can be formulated thus:

a finite group in which for each p every cyclic p-subgroup is pronormal is a soluble $\mathfrak{J}$-group. On the other hand it is known that in a soluble $\mathfrak{J}$-group all subgroups are pronormal [**5**].

[3] This property may be combined with the well-known theorem of Frobenius on p-nilpotent groups [**7**, Theorem iv. 5.c] to give another proof of Theorem 2.

The above corollary allows us to draw a further conclusion from Theorem 3.

COROLLARY (TO THEOREM 3). *A finite group which satisfies $\mathfrak{C}_p$ and is p-soluble has p-length $\leqq 1$.*

For if G is such a group every subgroup of G is either p-nilpotent or p-perfect and this excludes the possibility of a subgroup of p-length 2. (This result also follows from the main theorem of [1].)

On the other hand such a group need not be p-nilpotent, as the symmetric group of degree 4 with $p=3$ shows.

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