THE QUATERNION GROUP IN PLANE GEOMETRY

K. SITARAM

Introduction. Consider an affine Desarguian plane $\wp$, whose coordinate ring is k. We know that k is a division ring. (See, for instance, [**1**, Chapter II, §§1–4, pp. 51–66].) Let k^* denote the multiplicative group of the nonzero elements of k. In this paper we give a simple geometric configuration which exists in $\wp$ iff the quaternion group, which we denote by Q, is a subgroup of k^*. We also obtain some consequences of the existence of this configuration.

It is easy to see that if Q is a subgroup of k^*, then the characteristic of k is different from 2 and hence we can speak of harmonic conjugation and in particular of the middle point of a pair of points in $\wp$. (See [**1**, pp. 79–85].)

1. **Theorem.** *Q is a subgroup of k^* iff there are three distinct pairs of collinear points P_i, P_i', $(i=1, 2, 3)$ in $\wp$ having the same middle point 0 and such that P_i, P_i' are harmonically conjugate with respect to P_j, P_j', $i \neq j$.*

PROOF. Suppose $Q=\{\pm 1, \pm\sigma, \pm\tau, \pm\sigma\tau \mid \sigma^2=\tau^2=-1, \tau\sigma=-\sigma\tau\}$ is a subgroup of k^*. On the line $y=0$ let us take 0 to be (0, 0) and $P_1(1, 0)$, $P_1'(-1, 0)$; $P_2(\sigma, 0)$, $P_2'(-\sigma, 0)$; $P_3(\tau, 0)$, $P_3(-\tau, 0)$. We know that the harmonic conjugate of $(\sigma, 0)$ with respect to $(1, 0)$, $(-1, 0)$ is $(\sigma^{-1}, 0)$ and since $\sigma^2=-1$, we have $\sigma^{-1}=-\sigma$. i.e. P_2, P_2' are harmonic conjugates with respect to P_1, P_1'; so are P_3, P_3'. The harmonic conjugate of P_2 with respect to P_3, P_3' is $[\tau-(-\tau+(\sigma-\tau)^{-1})^{-1}, 0]$ by formula 2.9 of [**1**]. We can easily check that

$$(\sigma-\tau)^{-1}=-(\sigma-\tau)/2 \quad \text{and} \quad (\sigma+\tau)^{-1}=-(\sigma+\tau)/2.$$

Using these relations we get the coordinates of the harmonic conjugate of P_2 to be $(-\sigma, 0)$ i.e. P_2, P_2' are indeed harmonic conjugates with respect to P_3, P_3', which proves the first part of our theorem.

Let us now prove the converse. Without loss of generality we can take the given line as $y=0$, 0 as (0, 0) and P_1 as (1, 0). Let P_2 and P_3 be $(\alpha, 0)$ and $(\beta, 0)$. P_1', P_2' and P_3' will then be $(-1, 0)$, $(-\alpha, 0)$ and $(-\beta, 0)$. Since $(\alpha^{-1}, 0)$ is the harmonic conjugate of $(\alpha, 0)$ with respect to $(1, 0)$, $(-1, 0)$, we must have $\alpha^{-1}=-\alpha$ or $\alpha^2=-1$. Similarly $\beta^2=-1$.

P_2' being the harmonic conjugate of P_2 with respect to P_3, P_3' we have $-\alpha=\beta-(-\beta+(\alpha-\beta)^{-1})^{-1}$, which simplifies to $\beta\alpha=-\alpha\beta$.

Received by the editors April 25, 1966.

Hence $Q=\{\pm 1,\ \pm\alpha,\ \pm\beta,\ \pm\alpha\beta \mid \alpha^2=\beta^2=-1,\ \beta\alpha=-\alpha\beta\}$ is a subgroup of k^*.

2. **Consequences of the theorem.** When Q is a subgroup of k^*, it follows of course from the theorem that *every* line in $\wp$ has three pairs of points with a common middle point satisfying the harmonic property. Now let $\wp'$ be the corresponding projective plane obtained by adjoining "the line at infinity" to $\wp$. Our theorem becomes for $\wp'$:

Q is a subgroup of k^* iff in the projective plane $\wp'$ every line contains four pairs of points such that each pair is harmonically conjugate with respect to every other pair.

We now give the following two corollaries, which classify projective Desarguian planes on the basis of the harmonic property. The first one can easily be checked:

COROLLARY 1. *-1 $(\neq 1)$ is a square in k^* iff every line of $\wp'$ contains three pairs of points satisfying the harmonic property.*

Let us next prove

COROLLARY 2. *If in $\wp'$ there are four pairs of points on a line satisfying the harmonic property, then there exists one and only one more such pair on that line. Thus in any projective Desarguian plane there cannot be more than five pairs of points on a line satisfying the harmonic property.*

PROOF. Consider the affine plane $\wp$ with points P_i, P_i' $(i=1, 2, 3)$ as in the theorem. It is easy to see that $P_4(\sigma\tau, 0)$, $P_4'(-\sigma\tau, 0)$ is a pair of points with 0 as middle point and such that each of the pairs P_i, P_i' $(i=1, 2, 3)$ is harmonically conjugate with respect to P_4, P_4'. To complete the proof it is enough to show that if $(\rho, 0)$, $(-\rho, 0)$ is a pair of points harmonically conjugate with respect to each of P_i, P_i' $(i=1, 2, 3)$ then $\rho=\tau\sigma$ or $\sigma\tau$. Indeed, we have $\rho^2=-1$, $\rho\sigma=-\sigma\rho$ and $\rho\tau=-\tau\rho$. Using the associativity of multiplication in k^*, we get $(\sigma\tau\rho)^2=1$, so that $\sigma\tau\rho=1$ or -1 and hence $\rho=\tau\sigma$ or $\sigma\tau$.

Since Q is not commutative, it cannot be a subgroup of k^*, when k is finite. Hence the configuration of the theorem cannot occur in a finite plane. It would be of interest to have a geometric proof of this fact.

The author wishes to thank Professor Ernst Snapper for his encouragement while writing this paper.

REFERENCE

1. Emil Artin, *Geometric algebra*, Interscience, New York, 1957.

DARTMOUTH COLLEGE