# THE PRIME RADICAL IN A JORDAN RING

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There are several definitions of radicals for general nonassociative rings given in literature, e.g. [1], [2], and [5]. The $u$-prime radical of Brown-McCoy which is given in [2], is similar to the prime radical in an associative ring. However, it depends on the particular chosen element $u$. The purpose of this paper is to give a definition for the Brown-McCoy type prime radical for Jordan rings so that the radical will be independent from the element chosen.

Let $J$ be a Jordan ring, $x$ be an element in $J$; the operator $U_x$ is a mapping on $J$ such that $yU_x=2x\cdot(x\cdot y)-x^2\cdot y$ for all $y$ in $J$, or, equivalently, $U_x=2R_x^2-R_{x^2}$. If $A$, $B$ are subsets of $J$, $AU_B$ is the set of all finite sums of elements of the form $aU_b$, where $a$ is in $A$ and $b$ is in $B$.

LEMMA 1. *Let $P$ be a two sided ideal in $J$. Then the following three statements are equivalent.*

(a) *If $A$, $B$ are ideals in $J$ and $AU_B\subseteq P$, then either $A\subseteq P$ or $B\subseteq P$.*

(b) *If $A$, $B$ are ideals in $J$ with $A\cap c(P)\neq 0$ and $B\cap c(P)\neq 0$, then $AU_B\cap c(P)\neq 0$, where $c(P)$ is the complement of $P$.*

(c) *If $a$, $b$ are in $c(P)$, then $[a]\,U_{[b]}\cap c(P)\neq\emptyset$, where $[x]$ denotes the principal ideal in $J$ generated by $x$.*

PROOF. Obviously (a) and (b) are equivalent.

If (b) holds and $a, b\in c(P)$, then $[a]\cap c(P)\neq\emptyset$ and $[b]\cap c(P)\neq\emptyset$. Thus, $[a]U_{[b]}\cap c(P)\neq\emptyset$, i.e. (c) holds.

If (c) holds and $A$, $B$ are ideals in $J$ such that $A\cap c(P)\neq\emptyset$ and $B\cap c(P)\neq\emptyset$, then there exists $a\in A\cap c(P)$ and $b\in B\cap c(P)$. Thus $[a]U_{[b]}\cap c(P)\neq\emptyset$. But $[a]\subseteq A$ and $[b]\subseteq B$, so $AU_B\cap c(P)\supseteq[a]U_{[b]}\cap c(P)\neq\emptyset$, i.e. (b) holds.

DEFINITION 1. An ideal $P$ in $J$ is called a prime ideal if it satisfies any one of the statements in the Lemma 1. A nonempty subset $M$ of $J$ is called a $Q$-system if whenever $A$, $B$ are ideals in $J$ such that $A\cap M\neq\emptyset$ and $B\cap M\neq\emptyset$ then $AU_B\cap M\neq\emptyset$.

If $P$ is an ideal in $J$, then $c(P)=M$ is a $Q$-system if, and only if $P$ is a prime ideal.

DEFINITION 2. Let $A$ be an ideal in $J$, then $A^Q=\{r\in J\mid$ any $Q$-system in $J$ containing $r$ meets $A\}$ is called the $Q$-radical of $A$.

THEOREM 1. *If $A$ is an ideal in $J$, then $A^Q$ is the intersection of all the prime ideals $P^*$ in $J$ which contain $A$.*

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PROOF. If $b \in A^Q$ and $P^*$ is any prime ideal which contains $A$, then $b \in P^*$; otherwise, there exists a $Q$-system $c(P^*)$ containing $b$ which does not meet $A$, thus $b \notin A^Q$. Thus $A^Q \subseteq \bigcap P^*$.

Conversely, if $b \notin A^Q$, there exists a $Q$-system $M$ such that $b \in M$ and $M \cap A = \emptyset$. Applying Zorn's lemma to the family of all ideals in $J$ which contains $A$ but does not meet $M$, one finds a maximal element $P$ (partial ordering being taken as the usual set inclusion). Since $b$ is in $M$, $b$ is not in $P$. Thus it remains to show that $P$ is a prime ideal.

If $B$, $C$ are ideals in $J$ such that $B \not\subseteq P$ and $C \not\subseteq P$ then both $B+P$, and $C+P$ meet $M$. Thus $(P+B)U_{(P+C)}$ meets $M$. But $(P+B)U_{(P+C)} \subseteq BU_C+P$, thus $BU_C \not\subseteq P$. Hence $P$ is prime.

DEFINITION 3. An ideal $P$ in $J$ is a semiprime ideal if for any ideal $A$ in $J$, $AU_A \subseteq P$ implies $A \subseteq P$. A nonempty subset $M$ of $J$ is called a $SQ$-system if for any ideal $A$ in $J$ such that $A \cap M \neq \emptyset$, then $AU_A \cap M \neq \emptyset$.

The proof of Lemma 1 can be easily applied here to show an ideal $P$ in $J$ is semiprime if, and only if, one of the following statements holds.

(a) If $A$ is an ideal such that $A \cap c(P) \neq \emptyset$, then $AU_A \cap c(P) \neq \emptyset$.

(b) If $a \in c(P)$ then $[a]U_{[a]} \cap c(P) \neq \emptyset$.

If $P$ is an ideal in $J$, then $c(P)$ is a $SQ$-system if, and only if, $P$ is semiprime.

DEFINITION 4. Let $A$ be an ideal in $J$, the set $A_Q = \{r \in J \mid$ any $SQ$-system containing $r$ meets $A\}$ is called the $SQ$-radical of $A$.

THEOREM 2. *Let $A$ be an ideal in $J$, then the following statements hold*

(a) $A_Q = \bigcap P_*$, *where $P_*$ are taken from all semiprime ideals in $J$ which contain $A$.*

(b) *$A_Q$ is a semiprime ideal.*

(c) *$A$ is semiprime if, and only if, $A = A_Q$.*

PROOF. (a) If $x \in A_Q$ and $P_*$ is a semiprime ideal in $J$ containing $A$, then $x \in P_*$; otherwise, $c(P_*)$ is a $SQ$-system, contains $x$ but does not meet $A$, so $x \notin A_Q$. Thus $A_Q \subseteq \bigcap P_*$. Conversely, if $x \notin A_Q$, then there exists a $SQ$-system $M$ such that $x \in M$ and $M \cap A \neq \emptyset$. Applying Zorn's lemma to the family of ideals in $J$ containing $x$ but disjoint from $M$, one finds a maximal ideal $P_*$. It remains to show that $P_*$ is semiprime.

If $B$ is an ideal in $J$ such that $B \not\subseteq P_*$, then $P_*+B$ meets $M$. But $M$ is a $SQ$-system, thus $(P_*+B)U_{(P_*+B)}$ meets $M$. On the other hand, $(P_*+B)U_{(P_*+B)} \subseteq BU_B+P_*$, so $BU_B \not\subseteq P_*$.

(b) It follows from (a) that $A_Q$ is an ideal in $J$. If $B$ is an ideal in $J$ such that $BU_B \subseteq A_Q = \bigcap P_*$, then $B \subseteq P_*$ for all semiprime ideals $P_*$ containing $A$. Hence $B \subseteq \bigcap P_* = A_Q$. Thus $A_Q$ is a semiprime.

(c) Since $A_Q$ is a semiprime ideal, it is the smallest semiprime ideal in $J$ containing $A$. Thus $A$ is semiprime if, and only if, $A = A_Q$.

LEMMA 2. *Let a be an element in J and S is a SQ-system in J containing a. Then there exists a Q-system M such that a is in M and $M \subseteq S$.*

PROOF. We first construct a sequence $M = \{a_1, a_2, \cdots, a_n, \cdots\}$ of elements of $J$ where $a_1 = a$, $a_2 \in [a_1] U_{[a_1]} \cap S, \cdots, a_{k+1} \in [a_k] U_{[a_k]} \cap S, \cdots$. Clearly, $a \in M$ and $M \subseteq S$. It remains to show that $M$ is a $Q$-system, i.e. $[a_i] U_{[a_j]} \cap S \neq \emptyset$, for all $i, j$.

Note that $a_{i+1} \in [a_i]$, so $[a_{i+1}] \subseteq [a_i]$ and hence $[a_j] \subseteq [a_i]$ if $j \geq i$. If we let $K$ be the larger of $i$ and $j$ then $a_{k+1} \in [a_k] U_{[a_k]} \cap S \subseteq [a_i] U_{[a_j]} \cap S$.

THEOREM 3. *For any ideal A in J, $A^Q = A_Q$. $A^Q$ is called the prime radical of the ideal A.*

PROOF. Since every prime ideal is a semiprime ideal, it is clear that $A^Q = \bigcap P^* \supseteq \bigcap P_* = A_Q$.

Conversely, if $x \in A^Q$, and $S$ is a $SQ$-system containing $x$, then by Lemma 2, there exists a $Q$-system $M$ such that $x \in M$ and $M \subseteq S$ since $M$ meets $A$, $S$ meets $A$ also.

DEFINITION 5. The prime radical, $R(J)$, of a Jordan ring $J$ is the prime radical of the zero ideal in $J$. A Jordan ring is $Q$-semisimple if and only if $R(J) = (0)$.

THEOREM 4. *Let J be a Jordan ring and R(J) be the prime radical of J, then $R(J/R(J)) = (0)$, i.e. $J/R(J)$ is a Q-semisimple ring.*

PROOF. Let $\theta: a \rightarrow \bar{a}$ be the natural homomorphism from $J$ onto $J/R(J) = \bar{J}$. It is easy to check that the image of any prime ideal in $J$ is a prime ideal in $\bar{J}$. Let $\bar{a} \in R(\bar{J})$ and $P$ be any prime ideal in $J$. Then $\bar{a} \in \bar{P} = P/R(J)$. Hence, $a \in \theta^{-1}(\bar{P}) = P$, so $a \in \bigcap P = R(J)$ and $\bar{a} = 0$.

DEFINITION 6. A ring $J$ is a prime ring if, and only if, (0) is a prime ideal in $J$.

Thus, a prime ring must be $Q$-semisimple, and an ideal $P$ in $J$ is prime if, and only if, $J/P$ is a prime ring.

As in the case of associative rings, one can easily prove the following two assertions.

(a) A ring $R$ is a subdirect sum of $S_i$, $i \in I$ if, and only if, for each $i \in I$, there exists a homomorphism $\phi_i$ from $R$ onto $S_i$ and that $0 \neq r \in R$ implies $\phi_i(r) \neq 0$ for at least one $i \in I$.

(b) A ring is a subdirect sum of rings $S_i$, $i \in I$, if, and only if, for each $i \in I$ there exists a two sided ideal $K_i$ in $R$ such that $R/K_i \cong S_i$ and $\bigcap K_i = (0)$.

We obtain the following two theorems. The proof is similar to that in the associative case.

THEOREM 5. *A necessary and sufficient condition that a Jordan ring be isomorphic to a subdirect sum of prime rings is that J is Q-semisimple.*

In the presence of the descending chain condition on ideals in $J$, one may choose a finite subset of prime ideals $\{P_i \mid i=1, \cdots, n\}$ in $J$ such that $\bigcap P_i = 0$ and $\bigcap_{i\neq j} P_i \neq 0$ for any $j=1, 2, \cdots, n$.

THEOREM 6. *If J is a Jordan ring with descending chain condition on prime ideals then J is Q-semisimple if, and only if, Q is a full direct sum of finite numbers of prime ideals in J.*

THEOREM 7. *Let A be an ideal in Jordan ring J and $r \in A_Q$, then there exists a positive integer k such that $r^k \in A$.*

PROOF. It is sufficient to show that if $r \in A_Q$, then the set $M = \{r, r^3, r^{3^2}, \cdots, r^{3^k}, \cdots\}$ is a $SQ$-system.

Suppose $C$ is an ideal in $J$ and $r^{3^i} \in C \cap M$, then $r^{3^{i+1}} \in CU_C \cap M$. Thus $M$ is a $SQ$-system.

COROLLARY. *The prime radical of a Jordan ring J is a nilideal in J.*

PROOF. If $r \in R(J)$, then $r^k \in (0)$.

In a general nonassociative ring $R$, the nil radical $N(R)$ is the maximal nilideal in $R$[1]. As a consequence of the corollary, the prime radical of a Jordan ring is contained in the nil radical $N(J)$.

If $J$ is a finite dimensional Jordan algebra, every nilideal is a nilpotent ideal. Thus, $R(J)$ is contained in the classical radical $S(J)$, which is the maximal nilpotent ideal in $J$.

On the other hand, in the next theorem, any nilpotent ideal in $J$ is contained in $R(J)$. Thus, in this case, two definitions coincide. However, we are not sure whether in general this is also the case.

LEMMA 3. *Let A be an ideal in J. Then $A^3$ is an ideal of J and $A^3 = A\,U_A$.*

PROOF. The first assertion is a direct consequence of the linearized form of the Jordan identity: $[(a\cdot b)\cdot c]\cdot x = (a\cdot b)\cdot(c\cdot x) + (a\cdot c)\cdot(b\cdot x) + (b\cdot c)\cdot(a\cdot x) - [(a\cdot x)\cdot c]\cdot b - [(b\cdot x)\cdot c]\cdot a$. The second assertion is obtained from $4(x\cdot y)\cdot z = 2xU_{(y,z)} + 2yU_{(x,z)} = yU_{(x+z)} - yU_x - yU_z + xU_{(y+z)} - xU_y - xU_z \in A\,U_A$.

THEOREM 8. *A Jordan ring J is Q-semisimple if and only if it contains no nonzero nilpotent ideal.*

PROOF. By definition $S$ and part (c) of Theorem 2, $J$ is $Q$-semisimple if and only if $(0) = (0)_Q$. Thus $J$ being $Q$-semisimple is equivalent to the ideal (0) being semiprime. If $J$ contains a nonzero nilpotent ideal $M$ of nilindex $t$, then there exists a positive integer $t$ such that $M^{3^t}=0$ and $M^{3^{t-1}}\neq 0$. Thus (0) is not semiprime.

Conversely, if $J$ contains no nonzero nilpotent ideal and if (0) is not semiprime, then there exists a nonzero ideal $A$ such that $A U_A \subseteq 0$. Thus $A^3=0$ which is impossible.

COROLLARY. *The Q-radical $R(J)$ of a Jordan ring $J$ contains all the nilpotent ideals in $J$.*

PROOF. If $M$ is a nilpotent ideal in $J$, $\overline{M}$ is the image of $M$ under the natural homomorphism from $J$ onto $J/R(J)$. Since $\overline{M}$ is a nilpotent ideal in $\overline{J}$, $(\bar{0})$ is not a semiprime ideal in $\overline{J}$. If $\overline{A}$ is a nonzero ideal in $\overline{J}$ such that $\overline{A}^3=\overline{A}U_{\overline{A}}=(\bar{0})$, then $AU_A\subseteq R(J)$. But $R(J)$ is semiprime, so $A\subseteq R(J)$ and $\overline{A}=(\bar{0})$ which is a contradiction.

The following theorem is due to the referee.

THEOREM 9. *If a Jordan ring $J$ contains a maximal nilpotent ideal $S(J)$ then $R(J)=S(J)$.*

PROOF. Clearly $R(J)\supseteq S(J)$ by the corollary of Theorem 8. In the ring $\overline{J}=J/S(J)$ there are no nonzero nilpotent ideals by the maximality of $S(J)$. So $\overline{J}$ is $Q$-semisimple by Theorem 8.

If $r\in R(J)=\bigcap P^*$ then $r\in S(J)$. If $r\notin S(J)$, its image in $\overline{J}$ under the natural homomorphism would be $\bar{r}\neq\bar{0}$, so $\bar{r}\notin(\bar{0})=R(\overline{J})=\bigcap\overline{P}^*$ and $\bar{r}\notin\overline{P}^*$ for some prime ideal $\overline{P}^*$ in $J$. Let $P^*$ be the inverse image of $\overline{P}^*$ in $J$; then $\bar{r}\notin\overline{P}^*$ implies $r\notin P^*$. Since $r$ is in all prime ideals in $J$, $P^*$ cannot be prime. Thus there exists ideals $A, B$ in $J$ with $A\not\subseteq P^*$ and $B\not\subseteq P^*$ but $AU_B\subseteq P^*$. Passing to the homomorphic image $\overline{A}\subseteq\overline{P}^*$, $\overline{B}\subseteq\overline{P}^*$ but $\overline{A}U_{\overline{B}}\subseteq\overline{P}^*$. This contradicts the primeness of $\overline{P}^*$.

## REFERENCES


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