A PROPERTY OF ANGULAR CLUSTER SETS

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Let D denote the unit disk, let K denote the unit circle, and let f be a complex-valued function in D. If G is a subset of D, the cluster set $C_G(f, e^{i\theta})$ is defined as the set of all values w (including possibly $w=\infty$) for which there exists a sequence $\{z_n\}$ in G such that $z_n \to e^{i\theta}$ and $f(z_n)\to w$. In the case where $G=D$, $C_D(f, e^{i\theta})$ is usually denoted simply by $C(f, e^{i\theta})$. We will be particularly concerned with the cluster sets $C_{\Delta(\theta)}(f, e^{i\theta})$, where $\Delta(\theta)$ is a Stolz angle with vertex at $e^{i\theta}$, and with the outer angular cluster set $C_A(f, e^{i\theta})$, which is defined to be the union of all of the cluster sets $C_{\Delta(\theta)}(f, e^{i\theta})$.

Clearly $C_{\Delta(\theta)}(f, e^{i\theta}) \subset C_A(f, e^{i\theta})$, where the containment may, but need not be, proper. It is our purpose to show that this containment is actually an equality except on a subset of K of linear measure zero.

THEOREM 1. *Let f be an arbitrary complex-valued function in D. Then there exists a subset F of K, where F is a set of linear measure zero, such that for each point $e^{i\theta} \in K-F$ and each Stolz angle $\Delta(\theta)$ with vertex at $e^{i\theta}$,*

$$C_{\Delta(\theta)}(f, e^{i\theta}) = C_A(f, e^{i\theta}).$$

Theorem 1 can also be restated as follows.

THEOREM 1'. *Let f be an arbitrary complex-valued function in D. Then there exists a subset F of K, where F is a set of linear measure zero, such that for each point $e^{i\theta} \in K-F$ and each pair of Stolz angles $\Delta_1(\theta)$ and $\Delta_2(\theta)$ with vertex at $e^{i\theta}$,*

$$C_{\Delta_1(\theta)}(f, e^{i\theta}) = C_{\Delta_2(\theta)}(f, e^{i\theta}).$$

We remark that Theorem 1' is known in the case where f is a meromorphic function (see [**3**, Theorem 12, p. 68]).

To prove Theorem 1 we make use of the following lemma.

LEMMA. *Let f be an arbitrary complex-valued function in D, let α and β be two fixed real numbers satisfying $-\pi/2<\alpha<\beta<\pi/2$, and for each $e^{i\theta} \in K$ let*

$$\Delta(\theta) = \{z \in D: \alpha < \arg[1 - (z \mid e^{i\theta})] < \beta\}.$$

Then there exists a subset $F(\alpha, \beta)$ of K such that $F(\alpha, \beta)$ is a set of linear measure zero and for each $e^{i\theta} \in K - F(\alpha, \beta)$,

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$$C_{\Delta(\theta)}(f, e^{i\theta}) = C_A(f, e^{i\theta}).$$

PROOF. Let $\{V_n\}$ be a countable collection of open sets which form a base of open sets for the Riemann sphere W, and let $\{S_n\}$ be the collection of all finite unions of the sets V_n. For each positive integer j, let

$$\Delta(\theta, j) = \{z \in D: -\pi/2 + 1/j < \arg[1 - (z/e^{i\theta})] < \pi/2 - 1/j\}$$

and let $\Delta_r(\theta)$ denote the component of $\Delta(\theta) \cap \{z: |z| > r\}$ which has $e^{i\theta}$ as a boundary point. Finally, let

$$E(r, j, n) = \{e^{i\theta} \in K: f(\Delta_r(\theta)) \subset S_n \text{ and } C_{\Delta(\theta,j)}(f, e^{i\theta}) \text{ is not contained in } \overline{S}_n\}.$$

We claim that for each pair of positive integers j and n and each real number r with $0 < r < 1$, $E(r, j, n)$ is a set of linear measure zero.

Suppose that for some triple r, j, n, $E(r, j, n)$ is not a set of linear measure zero. Then, since $E(r, j, n)$ is a measurable subset of K, it has positive linear measure and hence there exists a perfect subset E^* of $E(r, j, n)$ such that E^* has positive measure. Let $D(r) = \{z: |z| \leqq r\}$ and let G be the union of $D(r)$ and all the sets $\Delta_r(\theta)$ for which $e^{i\theta} \in E^*$. Then, by a standard argument (see, for example [**3**, p. 71]), the boundary of G is a rectifiable Jordan curve. It follows that on a subset E' of E^*, where E' has positive linear measure, the boundary of G has a tangent at each point of E', and this tangent is the tangent to K at this point. For such a point $e^{i\theta} \in E'$ and for some $\epsilon > 0$,

$$\Delta(\theta, j) \cap \{z: |z - e^{i\theta}| < \epsilon\} \subset G.$$

But for each point of $G \cap \{z: |z| > r\}$, $f(z) \in S_n$. Thus $C_{\Delta(\theta,j)}(f, e^{i\theta}) \subset \overline{S}_n$, in violation of the definition of $E(r, j, n)$. Hence $E(r, j, n)$ must have linear measure zero.

If $C_{\Delta(\theta)}(f, e^{i\theta}) \neq C_A(f, e^{i\theta})$, then for some j, $C_{\Delta(\theta,j)}(f, e^{i\theta}) \neq C_{\Delta(\theta)}(f, e^{i\theta})$. Since each of these cluster sets is compact, there exists an integer n such that $C_{\Delta(\theta)}(f, e^{i\theta}) \subset S_n$ and $C_{\Delta(\theta,j)}(f, e^{i\theta})$ is not contained in $\overline{S}_n$. Hence for some real number r, $e^{i\theta} \in E(r, j, n)$. Let $F(\alpha, \beta)$ be the union of all the $E(r, j, n)$, where the union is taken over all rational numbers r between 0 and 1 and all pairs of positive integers n and j. Since $F(\alpha, \beta)$ is a countable union of sets of linear measure zero, it is itself a set of linear measure zero. Finally, if $e^{i\theta} \in K - F(\alpha, \beta)$, then $C_{\Delta(\theta)}(f, e^{i\theta}) = C_A(f, e^{i\theta})$. Thus the lemma is proved.

PROOF OF THEOREM 1. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of rational numbers satisfying $-\pi/2 < \alpha_n < \beta_n < \pi/2$ and such that for each pair of real numbers c and d satisfying $-\pi/2 < c < d < \pi/2$ there exists an integer n such that $c<\alpha_n<\beta_n<d$. Let $F=\bigcup_{n=1}^{\infty} F(\alpha_n, \beta_n)$. If $e^{i\theta}\in K-F$ and if $\Delta(\theta)$ is any Stolz angle with vertex at $e^{i\theta}$, there exists an integer n such that

$$\Delta'(\theta) = \{z \in D: \alpha_n < \arg[1 - (z/e^{i\theta})] < \beta_n\}$$

and $\Delta'(\theta)\subset\Delta(\theta)$. Since $e^{i\theta}\notin F(\alpha_n, \beta_n)$, we have $C_{\Delta'(\theta)}(f, e^{i\theta})=C_A(f, e^{i\theta})$ and thus $C_{\Delta(\theta)}(f, e^{i\theta})=C_A(f, e^{i\theta})$. Further, F is a countable union of sets of linear measure zero, and hence is itself a set of linear measure zero. Thus the theorem is proved.

By way of contrast to Theorem 1, we remark that both Collingwood [**1**, Theorem 4, p. 8] and Erdös and Piranian [**2**, Theorem 1, p. 155] have independently proved the following result.

THEOREM 2. *Let f be an arbitrary complex-valued function in D. Then there exists a subset E of K, where E is of first category, such that for each point $e^{i\theta}\in K-E$ and each Stolz angle $\Delta(\theta)$ with vertex at $e^{i\theta}$,*

$$C_{\Delta(\theta)}(f, e^{i\theta}) = C(f, e^{i\theta}).$$

We remark that functions exist for which the exceptional set F in Theorem 1 is not countable, as the following example shows.

For notational convenience we construct the desired function in the upper half plane U. Let R be the real line and let P denote the Cantor middle third set on the closed interval $[0, 1]$. Let $\{I_n\}$ be the collection of open intervals which are complementary to P in $(0, 1)$, and for each n let T_n be the triangular region bounded by the equilateral triangle in U, R having $\bar{I}_n$ as base. Let $T=\bigcup_{n=1}^{\infty} T_n$ and let $V=U-T$. Define f in U by

$$\begin{aligned} f(z) &= 0 \qquad \text{for } z \in V, \\ &= 1 \qquad \text{for } z \in T. \end{aligned}$$

For each $x_0\in P$ and each Stolz angle $\Delta(x_0)$ which meets the line $x=x_0$ and has angle opening less than $\pi/6$, $C_{\Delta(x_0)}(f, x_0)=\{0\}$. However, if $\Delta(x_0)$ is a Stolz angle at x_0 with angle opening greater than $2\pi/3$, $C_{\Delta(x_0)}(f, x_0)=\{0, 1\}$. Thus, if F is as in Theorem 1, $P\subset F$ and F is not countable. In fact, in this situation, F has positive capacity.

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