AN ISOMETRY OF H^p SPACES

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1. **Introduction.** In this note, we establish an isometry between $H^p(D)$ and $H^p(\mathfrak{D})$, where D is a generalized half-plane holomorphically equivalent to the bounded symmetric domain $\mathfrak{D}$. Our result is a direct generalization of one that is well known when D is the upper half-plane and $\mathfrak{D}$ the unit disk [2, p. 130], in which case $H^p(\mathfrak{D}) = \pi^{1/p}(1+z)^{2/p}H^p(D)$ in a sense which will be made clear. It is also an extension of a theorem of Koranyi [3, p. 344], which gives the desired result in the case $p=2$.

We begin with a brief recapitulation of the notation and some relevant results of the papers of Koranyi [3] and Stein [4]. $D = \{(z_1, z_2) \in V_1 \times V_2 : \operatorname{Im} z_1 - \Phi(z_2, z_2) \in \Omega\}$, where V_1 and V_2 are complex vector spaces of dimension n_1 and n_2; V_1 has a real form, Re V_1; $\Omega \subset \operatorname{Re} V_1$ is a domain of positivity; and Φ is a hermitian bilinear form on $V_2 \times V_2$ with respect to Re V_1 such that $\Phi_1(z, z_2) \in \bar{\Omega}$, $z_2 \in V_2$. The distinguished boundary of D is $B = \{(z_1, z_2) : \operatorname{Im} z_1 - \Phi(z_2, z_2) = 0\}$; B carries a natural measure, $d\beta$. There is a generalized Cayley transformation, $c: \mathfrak{D} \to D$. The distinguished boundary of $\mathfrak{D}$ is denoted by $\mathfrak{B}$, and carries a natural measure $d\mu$. Notice that we do not preclude the possibility that $V_2 = \{0\}$; in this case, D is a tube domain over the cone Ω. The spaces $H^p(D)$ and $H^p(\mathfrak{D})$ consist of all holomorphic functions on D (resp. $\mathfrak{D}$) satisfying:

$$\sup_{t \in \Omega} \int_B |f(u + (it, 0))|^p d\beta(u) < \infty$$

$$\left(\text{resp. } \sup_{r<1} \int_{\mathfrak{B}} |f(rv)|^p d\mu(v) < \infty\right), \quad p < \infty;$$

f bounded, $p = \infty$. It is shown in [3] and [4] that if $f \in H^p(D)$, $0 < p < \infty$, then $f_t(u) = f(u + (it, 0))$ converges in $L^p(B)$ as $t \to 0$ in Ω to a boundary function, denoted $f(u)$. Moreover, $f_t(u) \to f(u)$ a.e. on B if $t \to 0$ in Ω restrictedly, i.e., without coming too close to the boundary of Ω.

Finally, there is a Szegö kernel, $S_z(\zeta)$ defined for $z, \zeta \in D$ such that if $f \in H^2(D)$ then $f(z) = \int_B S_z(u) f(u) d\beta(u)$, and a Poisson kernel, $P_z(\zeta) = |S_z(\zeta)|^2 / S_z(z)$ such that if $f \in H^p(D)$, then $f(z) = \int_B P_z(u) f(u) du$, $1 \leqq p \leqq \infty$.

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2. **The isometry.**

DEFINITION. Let $f \in H^p(D)$, $0<p\leqq\infty$. Then Tf: $\mathfrak{D}\rightarrow \mathbf{C}$ is given by

$$Tf(w) = S_{ie}(ie)^{1/p} S_{ie}(cw)^{-2/p} f(cw).$$

Notice that since S_{ie} does not vanish on B, T can be extended to take the $L^p(B)$ boundary value of f onto a function defined on $c^{-1}(B)$, and hence almost everywhere on $\mathfrak{B}$. We denote this function also by Tf.

THEOREM. *T is an isometry from $H^p(D)$ onto $H^p(\mathfrak{D})$.*

(The isometry is one of Banach spaces in the case $1\leqq p\leqq\infty$. For $0<p<1$, T preserves the metric $\rho(f, g)=\int_B |f(u)-g(u)|^p d\beta(u)$ [resp. $\int_{\mathfrak{B}} |f(v)-g(v)|^p d\mu(v)$].)

Since the theorem is trivial for $p=\infty$, we assume from here on that $p<\infty$. The main argument of the theorem is contained in the following:

LEMMA. *Let $f\in H^p(D)$ and set $F(z)=f(z)S_{ie}(z)^{-2/p}$. Let $F(u)=f(u)S_{ie}(u)^{-2/p}$ be the boundary function defined almost everywhere on B. Then*

$$| F(z) |^p \leqq \int_B | F(u) |^p P_z(u) d\beta(u).$$

PROOF. We set

$$F_\epsilon(z) = F_\epsilon(z_1, z_2) = F(z_1, z_2) S_{ie}(\epsilon z_1, \epsilon^{1/2} z_2)^{2/p} S_{ie}(0)^{-2/p},$$

and notice that F_ϵ has a natural extension to a function defined a.e. on B, which we also call F_ϵ. It is immediate that $F_\epsilon(z)\rightarrow F(z)$ as $\epsilon\rightarrow 0$.

The Szegö kernel on D is given in terms of the norm function N on the tube domain T_Ω as follows [**3**, p. 338]:

$$S_{\zeta_1,\zeta_2}(z_1, z_2) = [N(-i(z_1 - \bar{\zeta}_1) - 2\Phi(z_2, \zeta_2))]^{-(n_1+n_2)/n_1}.$$

Therefore, for $\epsilon<1$,

$$\begin{aligned} | S_{ie}(\epsilon z_1, \epsilon^{1/2} z_2) | &= | N(1 - i\epsilon z_1) |^{-(n_1+n_2)/n_1} \\ &\leqq | N(\epsilon - i\epsilon z_1) |^{-(n_1+n_2)/n_1} = \epsilon^{-(n_1+n_2)} | S_{ie}(z_1, z_2) | , \end{aligned}$$

where the inequality follows from [**5**, Lemma 6.4] if we diagonalize $\operatorname{Im} z_1$, which lies in Ω.

We thus have $| F_\epsilon(z)| \leqq \epsilon^{-2(n_1+n_2)/p} |f(z)|$; in particular, $F_\epsilon \in H^p(D)$ for every ϵ.

But if g is any function in $H^p(D)$, then

$$(1) \qquad |g(z)|^p \leqq \int_B |g(u)|^p P_z(u)d\beta(u).$$

This is clear if $p \geqq 1$, since g is then the Poisson integral of its boundary value. And it is not hard to verify for arbitrary $p>0$. Briefly, one sets $g_\eta(z_1, z_2) = g(z_1 + i\eta e, z_2)$, noticing that g_η is then bounded, and considers $\tilde{g}_\eta(w) = g_\eta(cw) \in H^\infty(\mathfrak{D})$. It follows quickly by a method of Bochner [1] for dealing with H^p functions on circular domains that $|\tilde{g}_\eta(w)|^p \leqq \int_{\mathfrak{B}} |\tilde{g}_\eta(v)|^p \mathcal{P}_w(v)d\mu(v)$, where $\mathcal{P}$ is the Poisson kernel on $\mathfrak{D}$. But

$$\int_{\mathfrak{B}} |\tilde{g}_\eta(v)|^p \mathcal{P}_w(v)d\mu(v) = \int_{\mathfrak{B}} |g_\eta(u)|^p P_{cw}(u)d\beta(u)$$

[3, 4.1 and 4.3], and so the desired result follows if we let $\eta \to 0$ and recall that $g_\eta \to g$ in the $L^p(B)$ metric.

Finally, we notice that $|S_{ie}(\epsilon u_1, \epsilon^{1/2}u_2)| \leqq S_{ie}(0)$, and that $|S_{ie}(u)|$ and $|S_z(u)|$ are comparable when $z \in D$ is fixed. Thus

$$\begin{aligned} |F_\epsilon(u)|^p P_z(u) &\leqq A\,|F(u)|^p P_z(u) \\ &= AS_z(z)^{-1}|f(u)|^p |S_{ie}(u)|^{-2} |S_z(u)|^2 \leqq AA_z |f(u)|^p. \end{aligned}$$

If we now set $g = F_\epsilon$ in (1) and let $\epsilon \to 0$, we can apply the dominated convergence theorem to the RHS, completing the proof of the lemma.

To prove that T is an isometry from $H^p(D)$ into $H^p(\mathfrak{D})$ is now routine. We have

$$\begin{aligned} |Tf(w)|^p = S_{ie}(ie)\,|F(cw)|^p &\leqq S_{ie}(ie)\int_B |F(u)|^p P_{cw}(u)d\beta(u) \\ &= \int_{\mathfrak{B}} |Tf(v)|^p \mathcal{P}_w(v)d\mu(v). \end{aligned}$$

Since the Poisson integral is a convolution-type operator on $\mathfrak{D}$ [3, p. 344], it follows that $Tf \in H^p(\mathfrak{D})$ since $Tf(v) \in L^p(\mathfrak{B})$. Moreover,

$$\begin{aligned} \int_{\mathfrak{B}} |Tf(v)|^p d\mu(v) &= S_{ie}(ie)\int_{\mathfrak{B}} |f(cv)|^p |S_{ie}(cv)|^{-2} d\mu(v) \\ &= \int_{\mathfrak{B}} |f(cv)|^p P_{ie}(cv)^{-1} d\mu(v) = \int_B |f(u)|^p d\beta(u), \end{aligned}$$

which proves that T is an isometry.

We conclude the proof of the theorem by showing that the range of T includes the dense subset of $H^p(\mathfrak{D})$ consisting of functions h which are holomorphic on $\mathfrak{D}$ and continuous on $\overline{\mathfrak{D}}$. In fact, if we set $f(z) = h(c^{-1}z)S_{ie}(z)^{2/p}S_{ie}(ie)^{-1/p}$, then $h = Tf$, while f is the product of a bounded function and $S_{ie}(z)^{2/p} \in H^p(D)$.

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