A NOTE ON ASYMPTOTIC STABILITY

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1. **Introduction.** We consider a system of linear differential equations of the form

$$\dot{x} = [A + \epsilon B(t)]x \tag{1}$$

where x is an m-vector, A and $B(t)$ are complex $m\times m$ matrices, A is constant and skew-Hermitian ($A^* = -A$), B is continuous for all real t and of period $\omega > 0$, and ϵ is a small positive number. The problem of deciding the asymptotic behavior of the solutions of such a system is a common one in perturbation theory. The standard procedure for establishing the asymptotic stability of the trivial solution is to show that the characteristic multipliers have modulus less than 1 by expanding them in powers of ϵ [1]. If A and B depend on a number of parameters, as is usually the case, the computations may be formidable. However, it is possible to exploit the fact that A is skew-Hermitian to obtain a sufficient condition for asymptotic stability which considerably reduces the computational labor—in some cases to little more than the inspection of a certain Hermitian matrix.

We employ the following terminology. If $\phi(t)$, $W(t)$ are $m\times m$ matrices continuous for real t and $\dot{W}(t) = \phi(t)W(t)$, $W(0) = I$, we say that W is the fundamental matrix of $\dot{x} = \phi(t)x$. C^* denotes the adjoint (conjugate transpose) of C.

THEOREM. *Let A be skew-Hermitian and $B(t)$ continuous and of period $\omega > 0$. Let W_0 be the fundamental matrix of $\dot{x} = Ax$ and define $C_n = \int_0^{n\omega} W_0^*(t)B(t)W_0(t)dt$. If for some positive integer n, $C_n + C_n^*$ is negative definite, then for sufficiently small positive ϵ the trivial solution of* (1) *is asymptotically stable.*

2. **Lemmas.** All matrices will be m-square. $\|T\|$ is the operator norm of T,

$$\|T\| = \max\{\|Tx\|;\ \|x\| = 1\}, \quad \text{where } \|x\| = \left[\sum_{i=1}^{m} x_i\bar{x}_i\right]^{1/2}.$$

$\rho[T]$ is the maximum of the absolute values of the eigenvalues of T. Thus $\|T\|^2 = \rho[TT^*]$. We note that if W_0 is the fundamental matrix of $\dot{x} = Ax$, where $A^* = -A$, then $W_0^*(t) = W_0^{-1}(t)$. We choose to circumvent Floquet's theorem by means of the following trivial lemma.

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LEMMA 1. *Let W be the fundamental matrix of $\dot{x}=\phi(t)x$, ϕ continuous and of period $\omega>0$. $W(t)\to 0$ as $t\to\infty$ if (and only if) $\|W(n\omega)\|<1$ for some positive integer n.*

PROOF. $W(s+n\omega)=W(s)W^n(\omega)$ for all real s and integers n. Since any t can be written $t=s+n\omega$, $0\leqq s<\omega$, we have $W(t)=W(s)W^n(\omega)$, where $0\leqq s<\omega$. The condition of asymptotic stability of the trivial solution, $W(t)\to 0$ as $t\to\infty$, thus takes the form $W^n(\omega)\to 0$ as $n\to\infty$. But this is equivalent to $\|W^n(\omega)\|<1$ for some $n>0$. Since $W^n(\omega)=W(n\omega)$, the lemma is proved.

LEMMA 2. *Let $V(\epsilon)=I+\epsilon C+\epsilon^2R(\epsilon)$, where C is a matrix such that $C+C^*$ is negative definite, and $R(\epsilon)$ is continuous in ϵ. Then $\|V(\epsilon)\|<1$ for sufficiently small positive ϵ.*

PROOF. $\|V(\epsilon)\|^2=\rho[V(\epsilon)V^*(\epsilon)]$. Multiplying out yields $V(\epsilon)V^*(\epsilon)=I+\epsilon T(\epsilon)$, where $T(\epsilon)=C+C^*+\epsilon S(\epsilon)$, S being continuous and Hermitian. By continuity, there is $\epsilon_0>0$ such that $T(\epsilon)$ is negative definite for $|\epsilon|<\epsilon_0$. It follows that if $0<\epsilon<\epsilon_0$ and λ is an eigenvalue of $V(\epsilon)V^*(\epsilon)$, then $0\leqq\lambda<1$.

3. **Proof of the Theorem.** Let $W(t, \epsilon)$ be the fundamental matrix of (1) and write $W(t, \epsilon)=W_0(t)V(t, \epsilon)$. Then

$$V(t, \epsilon)=I+\epsilon\int_0^t W_0^{-1}(s)B(s)W_0(s)V(s, \epsilon)ds.$$

Substituting for V in the integrand and replacing W_0^{-1} by W_0^* yields

$$V(t, \epsilon) = I + \epsilon\int_0^t W_0^*(s)B(s)W_0(s)ds + \epsilon^2R(t, \epsilon),$$

where R is continuous. Let n be a positive integer for which $C_n+C_n^*$ is negative definite. Then by Lemma 2, $\|V(n\omega, \epsilon)\|<1$ for sufficiently small positive ϵ. Since W_0 is unitary, $\|W(t, \epsilon)\|=\|V(t, \epsilon)\|$. Lemma 1 concludes the proof.

The theorem is easily extended to the case that $B=B(t, \epsilon)$ provided B and $\partial B/\partial\epsilon$ are continuous for $|\epsilon|$ small. Except for the replacement of $B(t)$ by $B(t, 0)$ in the definition of C_n, the theorem and proof are unaltered.

The freedom offered by the arbitrariness of n can result in an appreciable computational simplification. If, for example, the off-diagonal entries of $C_n+C_n^*$ are bounded in n, while the diagonal entries tend to $-\infty$ with increasing n, the matrix is *obviously* negative definite for large n. Examples suggest that this situation is not unusual.

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Reference

1. I. G. Malkin, *Theory of stability of motion*, translated from Russian, AEC-tr-3352, U. S. Atomic Energy Commission, Oak Ridge, Tenn. p. 224. (Available from the Office of Technical Services, Department of Commerce, Washington, D. C.)

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