A NOTE ON ASYMPTOTIC STABILITY

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1. Introduction. We consider a system of linear differential equations of the form

\[ \dot{x} = [A + \epsilon B(t)]x \]

where \( x \) is an \( m \)-vector, \( A \) and \( B(t) \) are complex \( m \times m \) matrices, \( A \) is constant and skew-Hermitian \( (A^* = -A) \), \( B \) is continuous for all real \( t \) and of period \( \omega > 0 \), and \( \epsilon \) is a small positive number. The problem of deciding the asymptotic behavior of the solutions of such a system is a common one in perturbation theory. The standard procedure for establishing the asymptotic stability of the trivial solution is to show that the characteristic multipliers have modulus less than 1 by expanding them in powers of \( \epsilon \) [1]. If \( A \) and \( B \) depend on a number of parameters, as is usually the case, the computations may be formidable. However, it is possible to exploit the fact that \( A \) is skew-Hermitian to obtain a sufficient condition for asymptotic stability which considerably reduces the computational labor—in some cases to little more than the inspection of a certain Hermitian matrix.

We employ the following terminology. If \( \phi(t) \), \( W(t) \) are \( m \times m \) matrices continuous for real \( t \) and \( \dot{W}(t) = \phi(t)W(t) \), \( W(0) = I \), we say that \( W \) is the fundamental matrix of \( \dot{x} = \phi(t)x \). \( C^* \) denotes the adjoint (conjugate transpose) of \( C \).

Theorem. Let \( A \) be skew-Hermitian and \( B(t) \) continuous and of period \( \omega > 0 \). Let \( W_0 \) be the fundamental matrix of \( \dot{x} = Ax \) and define \( C_n = \int_0^\omega W_0^*(t)B(t)W_0(t)dt \). If for some positive integer \( n \), \( C_n + C_n^* \) is negative definite, then for sufficiently small positive \( \epsilon \) the trivial solution of (1) is asymptotically stable.

2. Lemmas. All matrices will be \( m \)-square. \( \| T \| \) is the operator norm of \( T \),

\[ \| T \| = \max \{ \| Tx \| ; \| x \| = 1 \}, \text{ where } \| x \| = \left( \sum_{i=1}^m x_i^2 \right)^{1/2}. \]

\( \rho[T] \) is the maximum of the absolute values of the eigenvalues of \( T \). Thus \( \| T \|^2 = \rho[T^*T] \). We note that if \( W_0 \) is the fundamental matrix of \( \dot{x} = Ax \), where \( A^* = -A \), then \( W_0^*(t) = W_0^{-1}(t) \). We choose to circumvent Floquet's theorem by means of the following trivial lemma.

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Lemma 1. Let $W$ be the fundamental matrix of $\dot{x} = \phi(t)x$, $\phi$ continuous and of period $\omega > 0$. $W(t) \to 0$ as $t \to \infty$ if (and only if) $\|W(n\omega)\| < 1$ for some positive integer $n$.

Proof. $W(s+n\omega) = W(s)W^n(\omega)$ for all real $s$ and integers $n$. Since any $t$ can be written $t = s + n\omega$, $0 \leq s < \omega$, we have $W(t) = W(s)W^n(\omega)$, where $0 \leq s < \omega$. The condition of asymptotic stability of the trivial solution, $W(t) \to 0$ as $t \to \infty$, thus takes the form $W^n(\omega) \to 0$ as $n \to \infty$. But this is equivalent to $\|W^n(\omega)\| < 1$ for some $n > 0$. Since $W^n(\omega) = W(n\omega)$, the lemma is proved.

Lemma 2. Let $V(\epsilon) = I + \epsilon C + \epsilon^2 R(\epsilon)$, where $C$ is a matrix such that $C + C^*$ is negative definite, and $R(\epsilon)$ is continuous in $\epsilon$. Then $\|V(\epsilon)\| < 1$ for sufficiently small positive $\epsilon$.

Proof. $\|V(\epsilon)\|^2 = \rho [V(\epsilon) V^*(\epsilon)]$. Multiplying out yields $V(\epsilon) V^*(\epsilon) = I + \epsilon T(\epsilon)$, where $T(\epsilon) = C + C^* + \epsilon S(\epsilon)$, $S$ being continuous and Hermitian. By continuity, there is $\epsilon_0 > 0$ such that $T(\epsilon)$ is negative definite for $|\epsilon| < \epsilon_0$. It follows that if $0 < \epsilon < \epsilon_0$ and $\lambda$ is an eigenvalue of $V(\epsilon) V^*(\epsilon)$, then $0 < \lambda < 1$.

3. Proof of the Theorem. Let $W(t, \epsilon)$ be the fundamental matrix of (1) and write $W(t, \epsilon) = W_0(t) V(t, \epsilon)$. Then

$$V(t, \epsilon) = I + \epsilon \int_0^t W_0^{-1}(s) B(s) W_0(s) V(s, \epsilon) ds.$$

Substituting for $V$ in the integrand and replacing $W_0^{-1}$ by $W_0^*$ yields

$$V(t, \epsilon) = I + \epsilon \int_0^t W_0^*(s) B(s) W_0(s) ds + \epsilon^2 R(t, \epsilon),$$

where $R$ is continuous. Let $n$ be a positive integer for which $C_n + C_n^*$ is negative definite. Then by Lemma 2, $\|V(n\omega, \epsilon)\| < 1$ for sufficiently small positive $\epsilon$. Since $W_0$ is unitary, $\|W(t, \epsilon)\| = \|V(t, \epsilon)\|$. Lemma 1 concludes the proof.

The theorem is easily extended to the case that $B = B(t, \epsilon)$ provided $B$ and $\partial B/\partial \epsilon$ are continuous for $|\epsilon|$ small. Except for the replacement of $B(t)$ by $B(t, 0)$ in the definition of $C_n$, the theorem and proof are unaltered.

The freedom offered by the arbitrariness of $n$ can result in an appreciable computational simplification. If, for example, the off-diagonal entries of $C_n + C_n^*$ are bounded in $n$, while the diagonal entries tend to $-\infty$ with increasing $n$, the matrix is obviously negative definite for large $n$. Examples suggest that this situation is not unusual.
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Reference


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