ON THE $L^1$ NORM AND THE MEAN VALUE OF A TRIGONOMETRIC SERIES

L. C. KURTZ AND S. M. SHAH

1. Introduction. In a recent paper [6] S. Uchiyama has derived lower bounds for

$$\int_0^1 \left| \sum_{k=1}^n \phi_k(t)c_k e(m_kx) \right| \, dx,$$

where $\phi_k$ is the $k$th Rademacher function, $\{m_k\}$ is a sequence of distinct integers, and $e(m_kx) = \exp(2\pi im_kx)$. Uchiyama's results hold except for values of $t$ in sets of arbitrarily small measure; these exceptional sets may include the values of $t$, near the origin, for which $\phi_k(t) = 1$. We find here lower bounds for

$$\int_0^1 \left| \sum_{k=1}^n c_k e(m_kx) \right| \, dx$$

which correspond to Uchiyama's results, but for these values of $t$. We have to put some conditions on the sequence $\{m_k\}$; simple examples show that our bounds cannot hold in general.

S. Chowla [1] conjectured that, for any sequence $\{m_k\}$ of increasing positive integers

$$(1) \quad \min_{0 \leq z < 1} \sum_{k=1}^n \cos 2\pi m_k x < -cn^{1/2}$$

for some absolute constant $c > 0$. S. Uchiyama [6] proved that given $n$ distinct integers $m_1, \ldots, m_n$, there exists always a subset $m_{j_1}, \ldots, m_{j_r}$ of $m_1, \ldots, m_n$, for which

$$\min_{0 \leq z < 1} \sum_{i=1}^r \cos 2\pi m_{j_i} x < -\frac{1}{4} \left( \frac{1}{6} \right)^{1/2} n^{1/2} = (-0.102 \ldots)n^{1/2}.$$

We prove here that if $\{m_k\}$ is an admissible sequence (for definition, see below), then

$$\min_{0 \leq z < 1} \sum_{k=1}^n \cos 2\pi m_k x < -\frac{1}{8} n^{1/2} = (-0.125)n^{1/2}.$$
2. Admissible sequences. We will say that a sequence \( \{m_k\} \) of positive integers is admissible provided that \( \{m_k\} \) is strictly increasing and \( m_k - m_j + m_l - m_p \neq 0 \) if \( k \neq j, k \neq p \) and \( j \neq l \) all hold. Note that this condition automatically holds if \( l = p \) since \( \{m_k\} \) is strictly increasing. Hence we shall assume \( l \neq p \). Similarly if \( k = l \) the condition is satisfied when \( j = p \). There are many sequences which are admissible, as the following lemma shows.

**Lemma.** If \( \{m_k\} \) is a sequence of positive integers such that \( m_{k+1} \geq m_k \geq 2 \), then \( \{m_k\} \) is admissible.

**Proof.** Let \( m_k = \max(m_k, m_j, m_p, m_l) \). Then
\[
m_k - m_j + m_l - m_p > m_k - m_j - m_p \geq m_k - 2 \max(m_j, m_p) \geq 0.
\]
If \( m_j = \max(m_k, m_j, m_p, m_l) \), then the same argument shows that
\[
m_k - m_j + m_l - m_p > 0.
\]

**Remarks.** For some deep results of Erdős, Chowla and others for similarly defined \( B_h \)-sequences \( \{m_k\} \), see [3, pp. 76–97].

3. Theorems. We let \( S_n(x) = \sum_{k=1}^{n} c_k e(m_k x) \), \( R_n = \sum_{k=1}^{n} |c_k|^2 \) and \( T_n = \sum_{k=1}^{n} |c_k|^4 \).

**Theorem 1.** If \( \{m_k\} \) is an admissible sequence, then
\[
\int_{0}^{1} |S_n(x)|^2 dx \geq \left( \frac{R_n}{2 - 1/n} \right)^{1/2}, \quad n = 1, 2, \ldots.
\]

**Proof.** It is easy to see that
\[
\int_{0}^{1} |S_n(x)|^2 dx = R_n.
\]

Also,
\[
\int_{0}^{1} |S_n(x)|^4 dx = \int_{0}^{1} \left\{ \sum_{k,j=1}^{n} c_k \bar{c}_j e((m_k - m_j)x) \right\}^2 dx
\]
\[
= \sum_{k,j=1}^{n} \left\{ \sum_{l,p=1}^{n} c_k \bar{c}_j c_l \bar{c}_p e((m_k - m_j + m_l - m_p)x) \right\} dx.
\]

We break this sum into three parts:

(A) The terms with \( k = j \) give
\[
\int_{0}^{1} \sum_{k=1}^{n} \left\{ \sum_{l,p=1}^{n} |c_k|^2 c_l c_p e(m_l - m_p)x \right\} dx = \sum_{k=1}^{n} \sum_{l=1}^{n} |c_k|^2 |c_l|^2.
\]

(B) When \( k \neq j \) we have some terms when \( k = p \) and \( j = l \). These terms give the sum \( \sum_{k,j=1}^{n} |c_k|^2 |c_j|^2 \).
(C) The remaining terms are given by

\[ \int_0^1 \sum_{k,j=1; k \neq j}^n \left\{ \sum_{l,p=1}^n c_k c_j c_l c_p e((m_k - m_j + m_l - m_p)x) \right\} \, dx \]

where either \( k \neq p \) or \( j \neq l \). In fact we may take \( k \neq p \) and \( j \neq l \) for if \( k \neq j \), \( k \neq p \), \( j \neq l \) in (C); so \( m_k - m_j + m_l - m_p \neq 0 \) since \( \{m_k\} \) is admissible, and the integral vanishes. Thus we have \( k^p \) and \( j^l \) for if \( k \neq p \) and \( j = l \), the corresponding integral vanishes. Thus we have \( k \neq j, k \neq p, j \neq l \) in (C); so \( m_k - m_j + m_l - m_p \neq 0 \) since \( \{m_k\} \) is admissible, and the integral vanishes. This shows that

\[ \int_0^1 |S_n(x)|^4 \, dx = \sum_{k,l=1}^n |c_k|^2 |c_l|^2 + \sum_{k,l=1; k \neq l}^n |c_k|^2 |c_l|^2 = 2R_n^2 - T_n. \]

Hölder's inequality now yields the result,

\[ \left( \int_0^1 |S_n(x)|^4 \, dx \right)^{2/3} \geq \left( \int_0^1 |S_n(x)|^2 \, dx \right)^{1/3} \left( \int_0^1 \left( \frac{|S_n(x)|^4 \, dx}{\left( \int_0^1 |S_n(x)|^2 \, dx \right)^{1/3}} \right) \right)^{2/3} \]

\[ = \frac{R_n/(2R_n^2 - T_n)^{1/2}}{2R_n^2 - T_n} \]

Hence

\[ \int_0^1 |S_n(x)| \, dx \geq \frac{R_n^{3/2}}{(2R_n^2 - T_n)^{1/2}} \geq \frac{R_n^{1/2}}{(2 - 1/n)^{1/2}}. \]

Note that there is an equality sign when \( n = 1 \). Also we have

\[ (2 - 1/n)^{-1/2} \leq \frac{||S_n||_\phi/||S_n||_2}{\leq 1}, \]

where \( ||S_n||_\phi (\phi = 1, 2) \) denotes \( L^\phi \) norm of \( S_n \).

We can get a similar result for real series

(3) \( T_n(x, \alpha) = T_n(x) = \sum_{k=1}^n \rho_k \cos 2\pi(m_kx + \alpha_k), \quad \rho_k \geq 0, \quad \alpha_k \text{ real.} \)

**Theorem 2.** If \( \{m_k\} \) is an admissible sequence, then

\[ \int_0^1 |T_n(x)| \, dx \geq \left( \sum_{k=1}^n \rho_k^2 \right)^{1/2} \]

\[ \cdot \frac{1}{2^{3/2}(2 - 1/n)^{1/2}}, \quad n = 1, 2, \ldots. \]

**Proof.** We have

\[ \int_0^1 |T_n(x)|^2 \, dx = \frac{1}{2} \sum_{k=1}^n \rho_k^2. \]

If \( U_n(x) = \sum_{k=1}^n \{\rho_k e(m_kx + \alpha_k)\} \), then \( T_n(x) = \text{Re} \, U_n(x) \). By Theorem 1,
\[
\int_0^1 |U_n(x)|^4 \, dx = 2 \left( \sum_{k=1}^n |\rho_k e(\alpha_k)|^2 \right)^2 - \sum_{k=1}^n |\rho_k e(\alpha_k)|^4
\]
\[
= 2 \left( \sum_{k=1}^n \rho_k^2 \right)^2 - \sum_{k=1}^n \rho_k^4.
\]

But \(|T_n(x)| \leq |U_n(x)|\) and H"older's inequality now gives
\[
\left( \int_0^1 |T_n(x)| \, dx \right)^{2/3} \leq \int_0^1 |T_n(x)|^2 \, dx \leq \left( \int_0^1 |U_n(x)|^4 \, dx \right)^{1/3}
\]
\[
\leq \frac{1}{2} \left( \sum_{k=1}^n \rho_k^2 \right)^{2/3} \leq \frac{1}{2} \frac{1}{(2 - 1/n)^{1/3}} \left( \sum_{k=1}^n \rho_k^2 \right)^{2/3}
\]
and \((4)\) is proved.

**Theorem 3.** If \(\{m_k\}\) is an admissible sequence, then

\[
\min_{0 \leq x \leq 1} T_n(x) \leq - \frac{1}{2^{5/2}(2 - 1/n)^{1/2}} \left( \sum_{k=1}^n \rho_k^2 \right)^{1/2}.
\]

**Proof.** Write

\[
T_n^+ = \max(T_n, 0), \quad T_n^- = - \min(T_n, 0).
\]

Then

\[
\int_0^1 |T_n| \, dx = \int_0^1 T_n^+ \, dx + \int_0^1 T_n^- \, dx,
\]
\[
\int_0^1 T_n \, dx = \int_0^1 T_n^+ \, dx - \int_0^1 T_n^- \, dx = 0.
\]

Hence

\[
2 \int_0^1 T_n^- \, dx = \int_0^1 |T_n(x)| \, dx.
\]

But \(T_n^- (x) \leq - \min_{0 \leq x \leq 1} T_n(x)\). Hence
\[
\frac{1}{2^{5/2}(2 - 1/n)^{1/2}} \left( \sum_{1}^{n} \rho_k \right)^{1/2} \leq \int_{0}^{1} T_{n}(x) \, dx \leq - \min_{0 \leq x \leq 1} T_{n}(x),
\]
and the theorem is proved.

**Corollary.** If \( \{m_k\} \) is an admissible sequence, then

\[
(6) \quad \min_{0 \leq x < 1} \sum_{1}^{n} \cos 2\pi m_k x \leq -2^{-5/2} \left( 2 - \frac{1}{n} \right)^{-1/2} n^{1/2} < -\frac{1}{8} n^{1/2}.
\]

We now consider this minimum when \( \{m_k\} \) is not necessarily an admissible sequence. Let \( \beta_0 \) be the unique root of the equation

\[
I(x) = \int_{0}^{3\pi/2} \cos u \frac{\cos u}{u^2} \, du = 0.
\]

The value of \( \beta_0 \) \((=.30844 \ldots)\) has been calculated (cf. [2], [4]) to fifteen places of decimal.

**Theorem 4.** Write

\[
(7) \quad m(n) = \min_{0 \leq x < 1} T_{n}(x, 0) = \min_{0 \leq x < 1} \sum_{k=1}^{n} \rho_k \cos 2\pi m_k x,
\]

and let \(0 < \beta < \beta_0, 1 \leq b < 1/(1-\beta), 0 < \gamma < 1 - b(1-\beta)\). Suppose that

\[
(8) \quad 1 \leq m_1 < m_2 < \cdots, \quad m_k < K k^{1/(1-\beta)} , \quad k = 1, 2, \cdots ,
\]

where \(K\) is a constant. Then

\[
(9) \quad \limsup_{n \to \infty} \left\{ -m(n)/n^\gamma \right\} > 0.
\]

We omit the proof which is similar to that of Theorem 3 of [2].

**Remarks.** A result similar to Theorem 4 can be proved for

\[
\min_{0 \leq x < 1} \sum_{1}^{n} \rho_k \cos (2\pi m_k x + 2\pi \alpha_k),
\]

provided we put a suitable condition on \(\alpha_k\) (cf. [2], [5]).

**Example.** Let \(1 \leq m_1 < m_2 < \cdots, m_k = O(k^{1+\varepsilon}), \rho_k > 0, k = 1, 2, \cdots ,\rho_k^{-1} = O(k^{\varepsilon})\) for every \(\varepsilon > 0\). Then the condition (8) is satisfied with \(b > 1\). Hence (9) holds with any number \(\gamma < \beta_0\). This extends a result of Chowla [1, p. 131].

**References**


University of Kentucky