ON THE L^1 NORM AND THE MEAN VALUE OF A TRIGONOMETRIC SERIES

L. C. KURTZ AND S. M. SHAH

1. Introduction. In a recent paper [6] S. Uchiyama has derived lower bounds for

$$\int_0^1 \left| \sum_{k=1}^{n} \phi_k(t) c_k e(m_k x) \right| dx,$$

where ϕ_k is the kth Rademacher function, $\{m_k\}$ is a sequence of distinct integers, and $e(m_k x) = \exp(2\pi i m_k x)$. Uchiyama's results hold except for values of t in sets of arbitrarily small measure; these exceptional sets may include the values of t, near the origin, for which $\phi_k(t) = 1$. We find here lower bounds for

$$\int_0^1 \left| \sum_{k=1}^{n} c_k e(m_k x) \right| dx$$

which correspond to Uchiyama's results, but for these values of t. We have to put some conditions on the sequence $\{m_k\}$; simple examples show that our bounds cannot hold in general.

S. Chowla [1] conjectured that, for any sequence $\{m_k\}$ of increasing positive integers

$$\min_{0 \leq x < 1} \sum_{k=1}^{n} \cos 2\pi m_k x < -cn^{1/2} \tag{1}$$

for some absolute constant $c > 0$. S. Uchiyama [6] proved that given n distinct integers $m_1, \cdots, m_n$, there exists always a subset $m_{j_1}, \cdots, m_{j_r}$ of $m_1, \cdots, m_n$, for which

$$\min_{0 \leq x < 1} \sum_{i=1}^{r} \cos 2\pi m_{j_i} x < -\frac{1}{4}\left(\frac{1}{6}\right)^{1/2} n^{1/2} = (-0.102 \cdots) n^{1/2}.$$

We prove here that if $\{m_k\}$ is an admissible sequence (for definition, see below), then

$$\min_{0 \leq x < 1} \sum_{k=1}^{n} \cos 2\pi m_k x < -\frac{1}{8} n^{1/2} = (-0.125) n^{1/2}.$$

Presented to the Society, September 1, 1967 under the title *On the L^1 norm and the mean value of a trigonometric polynomial*; received by the editors June 10, 1967.

2. **Admissible sequences.** We will say that a sequence $\{m_k\}$ of positive integers is admissible provided that $\{m_k\}$ is strictly increasing and $m_k - m_j + m_l - m_p \neq 0$ if $k \neq j$, $k \neq p$ and $j \neq l$ all hold. Note that this condition automatically holds if $l = p$ since $\{m_k\}$ is strictly increasing. Hence we shall assume $l \neq p$. Similarly if $k = l$ the condition is satisfied when $j = p$. There are many sequences which are admissible, as the following lemma shows.

LEMMA. *If $\{m_k\}$ is a sequence of positive integers such that $m_{k+1} | m_k \geqq 2$, then $\{m_k\}$ is admissible.*

PROOF. Let $m_k = \max(m_k, m_j, m_p, m_l)$. Then

$$m_k - m_j + m_l - m_p > m_k - m_j - m_p \geqq m_k - 2 \max (m_j, m_p) \geqq 0.$$

If $m_j = \max (m_k, m_j, m_p, m_l)$, then the same argument shows that $m_k - m_j + m_l - m_p \neq 0$.

REMARKS. For some deep results of Erdös, Chowla and others for similarly defined B_h-sequences ($h \geqq 2$), see [**3**, pp. 76–97].

3. **Theorems.** We let $S_n(x) = \sum_{k=1}^{n} c_k e(m_k x)$, $R_n = \sum_{k=1}^{n} |c_k|^2$ and $T_n = \sum_{k=1}^{n} |c_k|^4$.

THEOREM 1. *If $\{m_k\}$ is an admissible sequence, then*

$$\int_0^1 | S_n(x) | dx \geqq \left(\frac{R_n}{2 - 1/n} \right)^{1/2}, \qquad n = 1, 2, \cdots. \tag{2}$$

PROOF. It is easy to see that

$$\int_0^1 | S_n(x) |^2 dx = R_n.$$

Also,

$$\int_0^1 | S_n(x) |^4 dx = \int_0^1 \left\{ \sum_{k,j=1}^{n} c_k \bar{c}_j e((m_k - m_j)x) \right\}^2 dx$$

$$= \int_0^1 \sum_{k,j=1}^{n} \left\{ \sum_{l,p=1}^{n} c_k \bar{c}_j c_l \bar{c}_p e((m_k - m_j + m_l - m_p)x) \right\} dx.$$

We break this sum into three parts:

(A) The terms with $k = j$ give

$$\int_0^1 \sum_{k=1}^{n} \left\{ \sum_{l,p=1}^{n} | c_k |^2 c_l \bar{c}_p e(m_l - m_p)x \right\} dx = \sum_{k=1}^{n} \sum_{l=1}^{n} | c_k |^2 | c_l |^2.$$

(B) When $k \neq j$ we have some terms when $k = p$ and $j = l$. These terms give the sum $\sum_{k,j=1; k \neq j} |c_k|^2 |c_j|^2$.

(C) The remaining terms are given by

$$\int_0^1 \sum_{k,j=1;k\neq j}^{n} \left\{ \sum_{l,p=1}^{n} c_k\bar{c}_j c_l \bar{c}_p e((m_k - m_j + m_l - m_p)x) \right\} dx$$

where either $k\neq p$ or $j\neq l$. In fact we may take $k\neq p$ and $j\neq l$ for if $k\neq p$ and $j=l$, the corresponding integral vanishes. Thus we have $k\neq j$, $k\neq p$, $j\neq l$ in (C); so $m_k-m_j+m_l-m_p\neq 0$ since $\{m_k\}$ is admissible, and the integral vanishes. This shows that

$$\int_0^1 |S_n(x)|^4 dx = \sum_{k,l=1}^{n} |c_k|^2 |c_l|^2 + \sum_{k,l=1;k\neq l}^{n} |c_k|^2 |c_l|^2 = 2R_n^2 - T_n.$$

Hölder's inequality now yields the result,

$$\left(\int_0^1 |S_n(x)| dx\right)^{2/3} \geqq \int_0^1 |S_n(x)|^2 dx \Big/ \left(\int_0^1 |S_n(x)|^4 dx\right)^{1/3}$$
$$= R_n/(2R_n^2 - T_n)^{1/3}.$$

Hence

$$\int_0^1 |S_n(x)| dx \geqq R_n^{3/2}/(2R_n^2 - T_n)^{1/2} \geqq R_n^{1/2}/(2 - 1/n)^{1/2}.$$

Note that there is an equality sign when $n=1$. Also we have

$$(2 - 1/n)^{-1/2} \leqq \|S_n\|_1/\|S_n\|_2 \leqq 1,$$

where $\|S_n\|_p$ $(p=1, 2)$ denotes L^p norm of S_n.

We can get a similar result for real series

$$(3) \quad T_n(x, \alpha) = T_n(x) = \sum_{k=1}^{n} \rho_k \cos 2\pi(m_k x + \alpha_k), \quad \rho_k \geqq 0, \quad \alpha_k \text{ real.}$$

THEOREM 2. *If $\{m_k\}$ is an admissible sequence, then*

$$(4) \quad \int_0^1 |T_n(x)| dx \geqq \left(\sum_1^n \rho_k^2\right)^{1/2} \cdot \frac{1}{2^{3/2}(2 - 1/n)^{1/2}}, \qquad n = 1, 2, \cdots.$$

PROOF. We have

$$\int_0^1 |T_n(x)|^2 dx = \frac{1}{2} \sum_{k=1}^{n} \rho_k^2.$$

If $U_n(x) = \sum_{k=1}^n \{\rho_k\, e(m_k x + \alpha_k)\}$, then $T_n(x) = \text{Re } U_n(x)$. By Theorem 1,

$$\int_0^1 |U_n(x)|^4 dx = 2\left(\sum_{k=1}^n |\rho_k e(\alpha_k)|^2\right)^2 - \sum_{k=1}^n |\rho_k e(\alpha_k)|^4$$
$$= 2\left(\sum_{k=1}^n \rho_k^2\right)^2 - \sum_1^n \rho_k^4.$$

But $|T_n(x)| \leqq |U_n(x)|$ and Hölder's inequality now gives

$$\left(\int_0^1 |T_n(x)|\,dx\right)^{2/3}$$
$$\geqq \frac{\int_0^1 |T_n(x)|^2 dx}{\left(\int_0^1 |T_n(x)|^4 dx\right)^{1/3}} \geqq \frac{\int_0^1 |T_n(x)|^2 dx}{\left(\int_0^1 |U_n(x)|^4 dx\right)^{1/3}}$$
$$\geqq \frac{\frac{1}{2}\sum_{k=1}^n \rho_k^2}{\left[(2-1/n)^{1/3}\left(\sum_1^n \rho_k^2\right)^{2/3}\right]} = \left(\sum_{k=1}^n \rho_k^2\right)^{1/3} \frac{1}{2}\,\frac{1}{(2-1/n)^{1/3}},$$

and (4) is proved.

THEOREM 3. *If $\{m_k\}$ is an admissible sequence, then*

$$\min_{0\leqq x<1} T_n(x) \leqq -\frac{1}{2^{5/2}(2-1/n)^{1/2}}\left(\sum_1^n \rho_k^2\right)^{1/2}. \tag{5}$$

PROOF. Write

$$T_n^+ = \max(T_n, 0), \qquad T_n^- = -\min(T_n, 0).$$

Then

$$\int_0^1 |T_n|\,dx = \int_0^1 T_n^+ dx + \int_0^1 T_n^- dx,$$
$$\int_0^1 T_n dx = \int_0^1 T_n^+ dx - \int_0^1 T_n^- dx = 0.$$

Hence

$$2\int_0^1 T_n^- dx = \int_0^1 |T_n(x)|\,dx.$$

But $T_n^-(x) \leqq -\min_{0\leqq x\leqq 1} T_n(x)$. Hence

$$\frac{1}{2^{5/2}(2-1/n)^{1/2}}\left(\sum_{1}^{n}\rho_k^2\right)^{1/2} \leqq \int_0^1 T_n^-(x)dx \leqq -\min_{0\leq x\leq 1} T_n(x),$$

and the theorem is proved.

COROLLARY. *If $\{m_k\}$ is an admissible sequence, then*

$$\min_{0\leq x<1}\sum_{1}^{n}\cos 2\pi m_k x \leqq -2^{-5/2}\left(2-\frac{1}{n}\right)^{-1/2} n^{1/2} < -\frac{1}{8}n^{1/2}. \tag{6}$$

We now consider this minimum when $\{m_k\}$ is not necessarily an admissible sequence. Let β_0 be the unique root of the equation

$$I(x) = \int_0^{3\pi/2}\frac{\cos u}{u^x}du = 0.$$

The value of β_0 $(=.30844\cdots)$ has been calculated (cf. [2], [4]) to fifteen places of decimal.

THEOREM 4. *Write*

$$m(n) = \min_{0\leq x<1} T_n(x, 0) = \min_{0\leq x<1}\sum_{k=1}^{n}\rho_k\cos 2\pi m_k x, \tag{7}$$

and let $0<\beta<\beta_0$, $1\leqq b<1/(1-\beta)$, $0<\gamma<1-b(1-\beta)$. *Suppose that*

$$1\leqq m_1<m_2<\cdots,\qquad m_k<Kk^b\rho_k^{1/(1-\beta)},\qquad k=1,2,\cdots, \tag{8}$$

where K is a constant. Then

$$\limsup_{n\to\infty}\{-m(n)/n^\gamma\}>0. \tag{9}$$

We omit the proof which is similar to that of Theorem 3 of [2].

REMARKS. A result similar to Theorem 4 can be proved for

$$\min_{0\leq x<1}\sum_{1}^{n}\rho_k\cos(2\pi m_k x+2\pi\alpha_k),$$

provided we put a suitable condition on α_k (cf. [2], [5]).

EXAMPLE. Let $1\leqq m_1<m_2<\cdots$, $m_k=O(k^{1+\epsilon})$, $\rho_k>0$, $k=1,2,\cdots$, $\rho_k^{-1}=O(k^\epsilon)$ for every $\epsilon>0$. Then the condition (8) is satisfied with $b>1$. Hence (9) holds with any number $\gamma<\beta_0$. This extends a result of Chowla [1, p. 131].

REFERENCES

1. S. Chowla, *Some applications of a method of A. Selberg,* J. Reine Angew. Math. **217** (1965), 128–132.

2. J. Chidambaraswamy and S. M. Shah, *Trigonometric series with nonnegative partial sums*, J. Reine Angew. Math. **229** (1968), 163–169.

3. H. Halberstam and K. F. Roth, *Sequences*, Vol. I, Clarendon Press, Oxford, 1966.

4. R. D. Halbgewachs and S. M. Shah, *Trigonometric sums and Fresnel-type integrals*, Proc. Indian Acad. Sci. Sect. A **65** (1967), 227–232.

5. S. M. Shah, *Trigonometric series with non-negative partial sums*, Proc. Sympos. Pure Math., Vol. 11, Amer. Math. Soc., Providence, R. I., 1968.

6. S. Uchiyama, *On the mean modulus of trigonometric polynomials whose coefficients have random signs*, Proc. Amer. Math. Soc. **16** (1965), 1185–1190.

UNIVERSITY OF KENTUCKY