ON DISTRIBUTIONS WITH SUPPORT AT THE ORIGIN

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Among the results of the L. Schwartz theory of distributions is the following

THEOREM. *If α is a q-dimensional distribution whose support is the origin of R^q, then α is a linear combination of the Dirac δ-distribution and some of its derivatives.* [1].

It is the purpose of this paper to supply a new, sequential-theoretic, proof of this theorem. I should like to thank Professor Korevaar for suggesting this problem.

PROOF. Let $\langle \delta_N \rangle$ be a fundamental sequence of C^∞ functions which defines δ, and such that the support of δ_N is contained in

$$I_N = \{x = \langle x_1, \cdots, x_q \rangle \mid -1/N < x_i < 1/N, i = 1, \cdots, q\}, \qquad N = 1, 2, \cdots.$$

Then $\alpha_N = \alpha * \delta_N \in C^\infty$, the support of α_N is the same as that of δ_N, and $\alpha_N \to \alpha$ distributionally. Hence, for some multi-index $m = \langle m_1, \cdots, m_q \rangle$ there is a sequence $\langle G_N \rangle$ of C^∞ functions with the following properties:

(a) $D^m G_N = \alpha_N$ for each N; (b) $\langle G_N \rangle$ converges to a continuous function G uniformly on compact sets (and therefore $D^m G = \alpha$); and, because I_N supports α_N, (c) $G_N(x) = 0$ if, say, $x_i < -1$, $i = 1, \cdots, q$, $N = 1, 2, \cdots$.

Now let

$$P(x; t) = \prod_{i=1}^{q} \frac{(t_i - x_i)^{m_i - 1}}{(m_i - 1)!} \quad \text{if } x_i < t_i, i = 1, \cdots, q,$$
$$= 0 \qquad \text{otherwise,}$$

and

$$F_N(t) = \int_{-1}^{t_q} \cdots \int_{-1}^{t_1} D^m G_N(x) \cdot P(x; t) dx_1, \cdots, dx_q, \quad N = 1, 2, \cdots.$$

Integrating by parts m_i times with respect to x_i and observing that $D^r P(x; t) = 0$ if $r_i = m_i$ for some i, we have

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$$(1)\qquad F_N(t) = \sum_{0<r\leq m} (-1)^{r'} D^{m-r} G_N(x) \cdot D^{r'} P(x;t) \Big]_{x=\langle -1,\cdots,-1\rangle}^{x=\langle t_1,\cdots,t_q\rangle},$$

where $(-1)^r = \prod_{i=1}^q (-1)^{r_i}$ and $r' = \langle r_1 - 1, \cdots, r_q - 1\rangle$. Now, if $r \neq m$ then $D^{r'}P(t;t) = D^{m-r}G_N(-1, \cdots, -1) = 0$, and if $r = m$, then $D^{r'}P(t;t) = (-1)^{m'}$. Hence,

$$F_N(t) = (-1)^{m'}(-1)^{m'}G_N(t) = G_N(t).$$

Suppose now that $t_i < 0$ for some i. Then for all $N > -1/t_i$ we have $\alpha_N(x) = 0$ if $x \leqq t$ and $P(x;t) = 0$ otherwise. Hence, $F_N(t) \to 0$. On the other hand, if $t_i > 0$, $i = 1, \cdots, q$, then for all $N > \max_i \{1/t_i\}$ we have

$$F_N(t) = \int_{-1}^{1/N} \cdots \int_{-1}^{1/N} D^m G_N(x) \cdot P(x;t) dx_1 \cdots dx_q,$$

and replacing $\langle t_1, \cdots, t_q\rangle$ by $\langle 1/N, \cdots, 1/N\rangle$ in (1), we see that $F_N(t)$ is a polynomial in t of degree $< m_i$ in t_i, $N = 1, 2, \cdots$.

Since F_N converges to G uniformly on compact sets, it follows that G is equal to a *polynomial* $F(t)$ of degree $< m_i$ in t_i on $Q = \{t \mid t_i > 0, i = 1, \cdots, q\}$, and equal to zero on the complement of Q; in other words, $G = U \cdot F$, where U is the characteristic function of Q. Therefore,

$$(2)\qquad \alpha = D^m G = D^m(U \cdot F) = \sum_{0\leq r<m} \binom{m}{r} D^{m-r} U \cdot D^r F,$$

where

$$\binom{m}{r} = \prod_{i=1}^{q} \binom{m_i}{r_i}$$

and the strict inequality follows from the fact that $D^r F = 0$ if $r_i = m_i$ for some i. Now, since for $r < m$, $D^r F$ is a polynomial of degree less than $m_i - r_i$ in t_i, and $D^{m-r} U = D^{(m-r)'}\delta$, it follows that $D^{m-r} U \cdot D^r F$ is a constant multiple of $D^{(m-r')}\delta$. Thus, (2) may be rewritten as

$$(3)\qquad \alpha = \sum_{0\leq r_i<m_i} C_{(r_1,\cdots,r_q)} D^{(r_1,\cdots,r_q)}\delta.$$

Reference

1. L. Schwartz, *Théorie des distributions*. I, Actualités Sci. Indust. No. 1245, Hermann, Paris, 1957; p. 100.

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