THE PRIMITIVE SPECTRUM OF A TENSOR PRODUCT OF C^*-ALGEBRAS

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Let A and B be separable C^*-algebras, $A \odot B$ their involutive algebraic tensor product and $A \otimes B$ their C^*-tensor product. Let Prim A denote the primitive spectrum, i.e. the Jacobson structure space, of A, and let $\hat{A}$ denote the spectrum of A [1, 2.9.7, 3.1.5]. Denote the homeomorphism $(\pi, \nu) \mapsto \pi \otimes \nu$ of $\hat{A} \times \hat{B}$ into $(A \otimes B)^\wedge$ by γ [7]. We show that γ can be naturally projected to a homeomorphism of Prim $A \times$ Prim B onto Prim $A \otimes B$.

LEMMA 1. *Let $\mathfrak{d}$ be a primitive ideal of a C^*-algebra D. If the canonic homomorphism $D \to D/\mathfrak{d}$ is an isomorphism, then $\mathfrak{d} = 0$.*

PROOF. Evident.

PROPOSITION. *Every primitive ideal of $A \otimes B$ is of the form $\mathfrak{a} \otimes B + A \otimes \mathfrak{b}$, where $\mathfrak{a} \in$ Prim A, $\mathfrak{b} \in$ Prim B.*

PROOF. Let $z \in \mathfrak{c} \cap (A \odot B)$, where $\mathfrak{c}$ is a primitive ideal of $A \otimes B$. Suppose $\mathfrak{c} = \operatorname{Ker} \mu$, where μ is a factor representation of $A \otimes B$, and let μ_1, μ_2 be the restrictions of μ to A and B respectively [3]. Let $\mathfrak{a} = \operatorname{Ker} \mu_1$, $\mathfrak{b} = \operatorname{Ker} \mu_2$; since A and B are separable, $\mathfrak{a} \in$ Prim A and $\mathfrak{b} \in$ Prim B [2, p. 100]. So

$$z = \sum_{i=1}^{n} a_i \otimes y_i + \sum_{j=1}^{m} x_j \otimes b_j + \sum_{k=1}^{N} X_k \otimes Y_k,$$

where $a_i \in \mathfrak{a}$, $b_j \in \mathfrak{b}$; $x_j, X_k \in A$; $y_i, Y_k \in B$; $X_k \notin \mathfrak{a}$, $Y_k \notin \mathfrak{b}$. Thus

$$0 = \mu(z) = \mu\left(\sum_{k=1}^{N} X_k \otimes Y_k \right) = \sum_{k=1}^{N} \mu_1(X_k)\mu_2(Y_k).$$

Using [5, Theorem III], there exists a $N \times N$ matrix $(\alpha_{i,j})$ such that $\sum_{i=1}^{N} \alpha_{i,j}\mu_1(X_i) = 0$, $\sum_{j=1}^{N} \alpha_{i,j}\mu_2(Y_j) = \mu_2(Y_i)$. Thus $\sum_{k=1}^{N} X_k \otimes Y_k \in \mathfrak{a} \odot B + A \odot \mathfrak{b}$, and so $\mathfrak{c} \cap (A \odot B) = \mathfrak{a} \odot B + A \odot \mathfrak{b}$. It is easily seen that $\|\mu(\cdot)\|$ is a compatible norm on $(A \odot B)/(\mathfrak{a} \odot B + A \odot \mathfrak{b})$. Since the C^*-tensor product norm is the smallest compatible norm on $A \odot B$ [6], its quotient norm will be the smallest compatible norm on $(A \odot B)/(\mathfrak{a} \odot B + A \odot \mathfrak{b})$, and so $\|\mu(z)\| \geq \|\dot{z}\|$, where $z \mapsto \dot{z}$ denotes the canonic homomorphism $A \otimes B \to (A \otimes B)/(\mathfrak{a} \otimes B + A \otimes \mathfrak{b})$. The canonic homomorphism

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$$(A \otimes B)/(\mathfrak{a} \otimes B + A \otimes \mathfrak{b}) \to (A \otimes B)/\mathfrak{c}$$

$$= [(A \otimes B)/(\mathfrak{a} \otimes B + A \otimes \mathfrak{b})]/[\mathfrak{c}/(\mathfrak{a} \otimes B + A \otimes \mathfrak{b})]$$

is thus an isomorphism. By Lemma 1, $\mathfrak{c} = \mathfrak{a} \otimes B + A \otimes \mathfrak{b}$.

COROLLARY 1. *If π and ν are factor representations of A and B respectively, then* $\operatorname{Ker} \pi \otimes \nu = \operatorname{Ker} \pi \otimes B + A \otimes \operatorname{Ker} \nu$. *Thus* $(\operatorname{Ker} \pi, \operatorname{Ker} \nu) \mapsto \operatorname{Ker} \pi \otimes \nu$ *is a well-defined mapping of* $\operatorname{Prim} A \times \operatorname{Prim} B$ *onto* $\operatorname{Prim} A \otimes B$.

PROOF. The restrictions of $\pi \otimes \nu$ to A and B are π and ν respectively.

COROLLARY 2. *If $\mathfrak{a}$ and $\mathfrak{b}$ are primitive ideals of A and B respectively, then $\mathfrak{a} \otimes B + A \otimes \mathfrak{b}$ is a primitive ideal of $A \otimes B$.*

PROOF. Let π, ν be factor representations of A, B respectively such that $\mathfrak{a} = \operatorname{Ker} \pi$, $\mathfrak{b} = \operatorname{Ker} \nu$. Then $\mathfrak{a} \otimes B + A \otimes \mathfrak{b}$ equals $\operatorname{Ker} \pi \otimes \nu$ and is thus primitive.

COROLLARY 3 (CF. [8, PROPOSITION 1]). *If $\mathfrak{a}$ and $\mathfrak{b}$ are primitive ideals of A and B respectively, then $(A/\mathfrak{a}) \otimes (B/\mathfrak{b})$ is isomorphic to $(A \otimes B)/(\mathfrak{a} \otimes B + A \otimes \mathfrak{b})$.*

LEMMA 2. *The mapping* α: $(\mathfrak{a}, \mathfrak{b}) \mapsto \mathfrak{a} \otimes B + A \otimes \mathfrak{b}$ *is a bijection of* $\operatorname{Prim} A \times \operatorname{Prim} B$ *onto* $\operatorname{Prim} A \otimes B$.

PROOF. For $\mathfrak{b}, \mathfrak{b}' \in \operatorname{Prim} B$, $\mathfrak{a}, \mathfrak{a}' \in \operatorname{Prim} A$, $\mathfrak{a} \neq \mathfrak{a}'$, let $\mathfrak{a}' \not\ni x \in \mathfrak{a}$, $\mathfrak{b}' \not\ni y \notin \mathfrak{b}$. Injectivity follows since $\alpha(\mathfrak{a}', \mathfrak{b}') \not\ni x \otimes y \in \alpha(\mathfrak{a}, \mathfrak{b})$. Surjectivity follows from the proposition above.

THEOREM. *The mapping α is a homeomorphism of* $\operatorname{Prim} A \times \operatorname{Prim} B$ *onto* $\operatorname{Prim} A \otimes B$.

PROOF. The diagram

$$\begin{array}{ccc} A \times \hat{B} & \xrightarrow{\ \gamma\ } & \gamma(A \times \hat{B}) \\ \psi \big\downarrow & & \big\downarrow \text{Ker} \\ \operatorname{Prim} A \times \operatorname{Prim} B & \xrightarrow{\ \alpha\ } & \operatorname{Prim} A \otimes B \end{array}$$

is commutative, ψ denoting the canonic mapping.

REMARK. The C^*-algebra enveloping $\oplus_{\pi \in \hat{A}, \nu \in \hat{B}}\ \pi(A) \overline{\otimes}\ \nu(B)$, where $\pi(A) \overline{\otimes}\ \nu(B)$ is the algebra-of-operators tensor product of $\pi(A)$ and $\nu(B)$, has spectrum $\gamma(\hat{A} \times \hat{B})$. It defines a tensor product equivalent in general neither to $A \otimes B$ nor to the "projective" tensor product of Guichardet [4].

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