LINES IN A PLANAR SPACE

GRATTAN P. MURPHY¹

A planar space \( P \) is a set \( S \) together with a mapping \( A \) which attaches to each triple \( (p, q, r) \) of points of \( S \) a real number \( A(p, q, r) \) and which satisfies:

(i) If \( p=q \), then \( A(p, q, r) = A(p, r, q) = A(r, p, q) = 0 \) for every \( r \).
(ii) For every \( p, q, r, s \), \( A(p, q, r) = A(p, q, s) + A(p, r, s) + A(q, r, s) \).
(iii) For any \( p, q, r, s \); if \( A(p, q, r) = A(p, q, s) = 0 \), then \( p = q \) or \( A(q, r, s) = A(p, r, s) = 0 \).

For convenience we will write \( pqr \) for \( A(p, q, r) \) for the remainder of the paper.

The usual example of such a space is the Euclidean \( n \)-space with the \( A \)-function interpreted as the area of a triangle with vertices \( p \), \( q \), and \( r \).

Spaces satisfying (i) and (ii) and a variety of conditions in place of (iii) have been studied by Menger [6], Blumenthal [2], Froda [3], Gähler [5] and Freese and Andalafte [4].

For \( a \neq b \) points of \( P \) we define \( L[a, b] = \{ x \mid abx = 0 \} \). It follows readily that if \( L(a, b) \) and \( L(c, d) \) are distinct sets, then \( L(a, b) \cap L(c, d) \) contains at most one point.

If \( p \in P \) is not an element of \( L(a, b) \), we define a distance for points \( x, y \) of \( L \) by setting \( d(x, y) = pxy \).

If \( x = y \), then \( pxy = 0 \), but \( d(x, y) = pxy = 0 \). If \( d(x, y) = 0 \), then \( pxy = 0 \) and, since \( x \) and \( y \) belong to \( L(a, b) \), \( xya = yxb = 0 \). Now, if \( x \neq y \), applying (iii) to the quadruple \( \{ p, x, y, a \} \) gives \( pxa = pya = 0 \). Application of (iii) to the quadruple \( \{ p, x, y, b \} \) gives \( pxb = pxb = 0 \). Then since \( pxa = pxb = 0 \), we have \( pab = xab = 0 \), also from (iii). However, \( pab > 0 \), since \( p \) is not in \( L(ab) \). Therefore, it must follow that \( x = y \).

Since condition (iii) may be variously applied to any three distinct points of \( S \) by letting \( s \) and another of the symbols \( p, q, r \) denote the same point, it follows that the \( A \)-function is symmetric. Symmetry of the distance function follows immediately. The tetrahedral inequality applied to \( \{ p, x, y, z \} \) gives \( pxy \leq pxz + pyz + xyz \). Since \( xyz = 0 \), we have \( d(x, y) \leq d(x, z) + d(y, z) \).

Consequently \( d(x, y) \) is a metric for \( L(a, b) \). The set \( L(a, b) \) with metric \( d \) is denoted \( M_p(a, b) \).

¹ This paper represents a portion of the author’s dissertation written under the direction of Raymond W. Freese at St. Louis University.

Presented to the Society, January 25, 1967; received by the editors June 10, 1967.

1106
We will utilize the following definitions.

A point $b$ is said to be between $a$ and $c$ (denoted by $B(a, b, c)$) iff $abc = 0$, $acx = abx + bxc$ for every $x$ and $a$, $b$ and $c$ are distinct.

A planar space $P$ is convex iff for each pair of different points $p$ and $q$ there exists a between point.

A sequence of points $\{x_n\}$ in a planar space $P$ has limit $x$ iff $\lim pxx_n = 0$ for every $p$ in $P$.

A sequence $\{x_n\}$ in a planar space $P$ is convergent with respect to $(a, b, c)$ iff $abc > 0$ and $\lim ax_i x_j = \lim bx_i x_j = \lim cx_i x_j = 0$.

A planar space is complete with respect to $(a, b, c)$ iff for every sequence $\{x_i\}$ convergent with respect to $(a, b, c)$, there exists a point $x$ of $P$ with $\lim x_i = x$.

**Theorem.** If $P$ is a convex space which is complete with respect to $(p, a, b)$, then $M_p(a, b)$ is a complete, convex metric space.

If $x$ and $z$ are elements of $M_p(a, b)$, then they are elements of $P$ also. From convexity, there exists a $y$ in $P$ such that $B(x, y, z)$ holds. This gives $xyz = 0$, so that $y$ is in $M_p(a, b)$, and $pxy + pyz = pzx$ which results in $d(x, y) + d(y, z) = d(x, z)$. But, then $y$ is a between point of $x$ and $z$, so that $M_p(a, b)$ is convex. If $\{x_n\}$ is a convergent sequence in $M_p(a, b)$, then $\lim d(x_i, x_j) = 0$. But this implies that $\lim pxx_j = 0$. Then, since $ax_i x_j = bx_i x_j = 0$ and $pab \neq 0$, $\{x_n\}$ is a convergent sequence with respect to $(p, a, b)$. But $P$ is complete with respect to $(p, a, b)$ so there is an $x$ in $P$ which is the limit of $\{x_n\}$. From $abx_i = 0$ for every $i$, it follows that $abx = 0$ and that $x$ is in $M_p(a, b)$, which is, therefore, complete.

A subset $S$ of a planar space $P$ is said to be $A$-congruent with a subset $S'$ of a planar space $P'$ (denoted $S \equiv S'$) iff there exists a 1-1 mapping of $S$ onto $S'$ such that $pqr = p'q'r'$, where $p'$, $q'$, $r'$ are the images of $p$, $q$, and $r$.

**Corollary.** If each four points of $P$ are $A$-congruent with four points of $E_3$, then $M_p(a, b)$ is congruently contained in $E_3$.

Let $x$, $y$, $z \in L(a, b)$ and be distinct points. Then $xyz = 0$. If each of the four points of $P$ are $A$-congruently contained in $E_3$, then there exist $p'$, $x'$, $y'$, and $z'$ in $E_3$ with $p'x'y' = pxy$, $p'x'z' = pzx$, $p'y'z' = pyz$ and $x'y'z' = xyz = 0$. Consequentially $x'$, $y'$, and $z'$ are collinear and $p'$, $x'$, $y'$, and $z'$ are coplanar. It follows that one of the points $x'$, $y'$, or $z'$ is a between point of the other two. Let $y'$ be the between point. Then $p'x'z' = p'x'y' + p'y'z'$ from which $pxz = pxy + pyz$ follows. But this gives $d(x, z) = d(x, y) + d(y, z)$ so that the three points are embed-
dable in $E_1$. There are more than four distinct points in $M_p(a, b)$ since $a \neq b$ and $M_p(a, b)$ is convex.

Since every semimetric space containing more than four points and having the property that each three of its points are embeddable in $E_1$ is embeddable in $E_1$ [1], $M_p(a, b)$ is congruently contained in $E_1$.

**Bibliography**

2. ———, *Distance geometries*, Univ. of Missouri Studies, 13, 1938, no. 2.

**University of Maine**