LINES IN A PLANAR SPACE

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A planar space P is a set S together with a mapping A which attaches to each triple (p, q, r) of points of S a real number $A(p, q, r)$ and which satisfies:

(i) If $p=q$, then $A(p, q, r)=A(p, r, q)=A(r, p, q)=0$ for every r.

(ii) For every p, q, r, s, $A(p, q, r) \leqq A(p, q, s)+A(p, r, s)+A(q, r, s)$.

(iii) For any p, q, r, s; if $A(p, q, r)=A(p, q, s)=0$, then $p=q$ or $A(q, r, s)=A(p, r, s)=0$.

For convenience we will write pqr for $A(p, q, r)$ for the remainder of the paper.

The usual example of such a space is the Euclidean n-space with the A-function interpreted as the area of a triangle with vertices p, q, and r.

Spaces satisfying (i) and (ii) and a variety of conditions in place of (iii) have been studied by Menger [**6**], Blumenthal [**2**], Froda [**3**], Gähler [**5**] and Freese and Andalafte [**4**].

For $a \neq b$ points of P we define $L[a, b]=\{x \mid abx=0\}$. It follows readily that if $L(a, b)$ and $L(c, d)$ are distinct sets, then $L(a, b) \wedge L(c, d)$ contains at most one point.

If $p \in P$ is not an element of $L(a, b)$, we define a distance for points x, y of L by setting $d(x, y)=pxy$.

If $x=y$, then $pxy=0$, but $d(x, y)=pxy=0$. If $d(x, y)=0$, then $pxy=0$ and, since x and y belong to $L(a, b)$, $xya=xyb=0$. Now, if $x \neq y$, applying (iii) to the quadruple $\{p, x, y, a\}$ gives $pxa=pya=0$. Application of (iii) to the quadruple $\{p, x, y, b\}$ gives $pxb=pyb=0$. Then since $pxa=pxb=0$, we have $pab=xab=0$, also from (iii). However, $pab>0$, since p is not in $L(ab)$. Therefore, it must follow that $x=y$.

Since condition (iii) may be variously applied to any three distinct points of S by letting s and another of the symbols p, q, r denote the same point, it follows that the A-function is symmetric. Symmetry of the distance function follows immediately. The tetrahedral inequality applied to $\{p, x, y, z\}$ gives $pxy \leqq pxz+pyz+xyz$. Since $xyz=0$, we have $d(x, y) \leqq d(x, z)+d(y, z)$.

Consequently $d(x, y)$ is a metric for $L(a, b)$. The set $L(a, b)$ with metric d is denoted $M_p(a, b)$.

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We will utilize the following definitions.

A point b is said to be between a and c (denoted by $B(a, b, c)$) iff $abc=0$, $acx=abx+bcx$ for every x and a, b and c are distinct.

A planar space P is convex iff for each pair of different points p and q there exists a between point.

A sequence of points $\{x_n\}$ in a planar space P has *limit* x iff $\lim pxx_n=0$ for every p in P.

A sequence $\{x_n\}$ in a planar space P is *convergent* with respect to (a, b, c) iff $abc>0$ and $\lim ax_ix_j=\lim bx_ix_j=\lim cx_ix_j=0$.

A planar space is complete with respect to (a, b, c) iff for every sequence $\{x_i\}$ convergent with respect to (a, b, c), there exists a point x of P with $\lim x_i=x$.

THEOREM. *If P is a convex space which is complete with respect to (p, a, b), then $M_p(a, b)$ is a complete, convex metric space.*

If x and z are elements of $M_p(a, b)$, then they are elements of P also. From convexity, there exists a y in P such that $B(x, y, z)$ holds. This gives $xyz=0$, so that y is in $M_p(a, b)$, and $pxy+pyz=pxz$ which results in $d(x, y)+d(y, z)=d(x, z)$. But, then y is a between point of x and z, so that $M_p(a, b)$ is convex. If $\{x_n\}$ is a convergent sequence in $M_p(a, b)$, then $\lim d(x_i, x_j)=0$. But this implies that $\lim px_ix_j=0$. Then, since $ax_ix_j=bx_ix_j=0$ and $pab\neq 0$, $\{x_n\}$ is a convergent sequence with respect to (p, a, b). But P is complete with respect to (p, a, b) so there is an x in P which is the limit of $\{x_n\}$. From $abx_i=0$ for every i, it follows that $abx=0$ and that x is in $M_p(a, b)$, which is, therefore, complete.

A subset S of a planar space P is said to be *A-congruent* with a subset S' of a planar space P' (denoted $S\equiv S'$) iff there exists a 1-1 mapping of S onto S' such that $pqr=p'q'r'$, where p', q', r' are the images of p, q, and r.

COROLLARY. *If each four points of P are A-congruent with four points of E_3, then $M_p(a, b)$ is congruently contained in E_1.*

Let $x, y, z\in L(a, b)$ and be distinct points. Then $xyz=0$. If each of the four points of P are A-congruently contained in E_3, then there exist p', x', y', and z' in E_3 with $p'x'y'=pxy$, $p'x'z'=pxz$, $p'y'z'=pyz$ and $x'y'z'=xyz=0$. Consequently x', y', and z' are collinear and p', x', y', and z' are coplanar. It follows that one of the points x', y', or z' is a between point of the other two. Let y' be the between point. Then $p'x'z'=p'x'y'+p'y'z'$ from which $pxz=pxy+pyz$ follows. But this gives $d(x, z)=d(x, y)+d(y, z)$ so that the three points are embeddable in E_1. There are more than four distinct points in $M_p(a, b)$ since $a \neq b$ and $M_p(a, b)$ is convex.

Since every semimetric space containing more than four points and having the property that each three of its points are embeddable in E_1 is embeddable in E_1 [1], $M_p(a, b)$ is congruently contained in E_1.

Bibliography

1. L. M. Blumenthal, *Theory and applications of distance geometry*, Clarendon Press, Oxford, 1953.

2. ———, *Distance geometries*, Univ. of Missouri Studies, 13, 1938, no. 2.

3. A. Froda, *Espaces p-metrique et leur topologie*, C. R. Acad. Sci. Paris **247** (1958), 849–852.

4. R. W. Freese and E. Z. Andalafte, *A characterization of 2-betweenness in 2-metric spaces*, Canad. J. Math. **18** (1966), 963–968.

5. S. Gähler, *2-metrische Räume und ihre topologische Struktur*, Math. Nachr. **26** (1963), 115–148.

6. K. Menger, *Untersuchungen über allgemeine Metrik*, Math. Ann. **100** (1928), 75–163.

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