ON THE EQUIVALENCE OF A TERNARY QUADRATIC FORM

E. H. HADLOCK AND AMIN MUWAFI

Introduction. In the classical theory of integral ternary quadratic forms a determinant

$$d = \begin{vmatrix} a & t & s \\ t & b & r \\ s & r & c \end{vmatrix} \neq 0$$

is associated with the form

$$f = ax^2 + by^2 + cz^2 + 2ryz + 2sxz + 2txy.$$

The g.c.d. of the cofactors of the elements of d is designated by Ω. Then Δ is defined by $d = \Omega^2\Delta$. The concepts of primitive, properly primitive and reciprocal forms as well as equivalence, properties and theorems relating to such concepts are given in [1].

LEMMA 1. *If g and its reciprocal G are primitive, positive or indefinite forms, then g is equivalent to a primitive form*

$$f = ax^2 + by^2 + cz^2 + 2ryz + 2sxz + 2txy \tag{1}$$

in which

$$(a, t) = 1 \quad and \quad s \neq 0. \tag{2}$$

PROOF. *Case* (i). *Let g and G be properly primitive.* By [2, Corollary, p. 16] g is equivalent to a form (1) whose first coefficient a and the third coefficient C of whose reciprocal form are positive integers which are relatively prime to each other and to $2\Omega\Delta$. Let p be an odd prime divisor of a. Then

$$(a, C) = (aC, 2\Omega\Delta) = 1 \quad \text{and} \quad \Omega C = ab - t^2 \Rightarrow p \nmid t. \tag{3}$$

Thus $(2)_1$ holds. If $s = 0$, then replacing x by $x + hz$, $h \neq 0$, in f gives a form in which the coefficient of $2xz \neq 0$ and such that a, t and C are unchanged.

Case (ii). *Let g be improperly primitive and G be properly primitive.* By [2, Theorem 20] we may assume that $a/2$ and C are odd and relatively prime to each other and to $\Omega\Delta$. Thus by $(3)_3$

$$\Omega C = (a/2)(2b) - t^2 \Rightarrow \text{as in Case (i)} \quad p \nmid t.$$

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Also we observe from the proof of [2, Theorem 19] that Ωt is odd. Hence $(2)_1$ holds.

Case (iii). *Let G be improperly primitive.* By [2, Theorem 21] we may assume that a and $C/2$ are odd relatively prime integers and prime to $\Delta\Omega$. Thus $(3)_3$ holds when C is replaced by $C/2$ and Ω by 2Ω and hence $(2)_1$ holds.

LEMMA 2. *There exists a linear transformation T which takes the form f of Lemma* 1 *into an equivalent form* $f_1 = ax^2 + b_1y^2 + c_1z^2 + 2r_1yz + 2s_1xz + 2t_1xy$ *with the same first coefficient a of f such that the coefficients* s_1 *and* t_1 *are relatively prime. Moreover the third coefficient* C_1 *in* F_1, *the reciprocal of* f_1, *is equal to C in F, the reciprocal of f.*

PROOF. Let T: replace x by $x+ky$, $y=y$, $z=z$. Then $s_1 = s \neq 0$, $t_1 = ak+t$. Also $C_1 = C$ for $\Omega C_1 = af(k, 1, 0) - t_1^2 = a(ak^2+b+2tk) - (ak+t)^2 = \Omega C$.

By (2) and Dirichlet's Theorem, there is an integer k such that $t_1 = ak+t = p$, a prime and $(t_1, s_1) = 1$ and where $s_1 \neq 0$.

THEOREM. *Every primitive, positive or indefinite, ternary quadratic form is equivalent to a primitive form in which the coefficients of* $2xz$ *and* $2xy$ *are one and zero respectively.*

PROOF. Apply the transformation T_1: $x=x$, replace y by $s_1y+\beta z$ and z by $-t_1y+\delta z$ to f_1 of Lemma 2 to obtain a form f_2 in which the coefficients s_2 of $2xz$ and t_2 of $2xy$ are given by

$$s_2 = s_1\delta + t_1\beta, \qquad t_2 = 0.$$

By Lemma 2, $(s_1, t_1) = 1$ so that there exist integers β and δ such that $s_2 = 1$. Moreover $|T_1| = s_1\delta + t_1\beta = s_2 = 1$.

REMARK. B. W. Jones has shown that every ternary quadratic form is equivalent to one in which the coefficient of $2xy$ is zero [3].

REFERENCES

1. L. E. Dickson, *Studies in the theory of numbers*, Univ. of Chicago Press, Chicago, Ill., 1930, pp. 6–10.

2. ———, ibid., Corollary, p. 16; Theorems, 19, 20 and 21, p. 17.

3. B. W. Jones, *The regularity of a genus of positive ternary quadratic forms*, Trans. Amer. Math. Soc., **33** (1931), 111–124.

4. L. E. Dickson, *Modern elementary theory of numbers*, Univ. of Chicago Press, Chicago, Ill., 1950; p. 90.

UNIVERSITY OF FLORIDA