ALGEBRAICALLY DEPENDENT FUNCTIONS OF A COMPLEX AND p-ADIC VARIABLE

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A set of functions $f_1(z), \cdots, f_m(z)$ are algebraically dependent if there is a polynomial $P(w_1, \cdots, w_m)$ with complex coefficients for which $P(f_1(z), \cdots, f_m(z)) \equiv 0$. In a previous paper (Hilliker [2]) conditions were established under which m functions $f_1(z), \cdots, f_m(z)$ of a complex variable z are algebraically dependent. It was assumed that $f_1(z), \cdots, f_m(z)$ are analytic at one point and that the functional values $f_\mu(z_\nu)$ $(\mu = 1, 2, \cdots, m; \nu = 1, 2, 3, \cdots)$ at a convergent sequence of distinct points $\{z_\nu\}$ are algebraic. Let $H(f_\mu(z_\nu))$ denote the height of the algebraic number $f_\mu(z_\nu)$; that is, the maximum of the absolute values of the (integral) coefficients of the minimal polynomial for $f_\mu(z_\nu)$. Let $\deg f_\mu(z_\nu)$ be the degree of $f_\mu(z_\nu)$. Then, roughly speaking, if $H(f_\mu(z_\nu))$ and $\deg f_\mu(z_\nu)$ for $\mu = 1, \cdots, m$ and $\nu = \nu_0 + 1, \nu_0 + 2, \cdots$, are not too large and if the sequence $\{z_\nu\}$ converges sufficiently rapidly, $f_1(z), \cdots, f_m(z)$ are algebraically dependent.

In this paper we wish to relax the restriction on the size of $H(f_\mu(z_\nu))$ and $\deg f_\mu(z_\nu)$ and on the rate of convergence of $\{z_\nu\}$. This is accomplished by assuming that the functions $f_1(z), \cdots, f_m(z)$ are simultaneously analytic in the complex and p-adic sense at some point. All of the theorems concerning algebraicness and algebraic dependence in the paper Hilliker [2] lend themselves to p-adic generalizations. To illustrate the type of extension that is possible we shall generalize Theorems IV and VII.

Let Q denote the field of rational numbers. We remind the reader of the following concepts. For $a/b \in Q$ we can write, for a given prime p, $a/b = p^\alpha a'/b'$ where a' and b' are not divisible by p and where α is an integer. The p-adic valuation of a/b is $|a/b|_p = p^{-\alpha}$. The completion of Q under this valuation is the p-adic number field Q_p which is analogous to the field of real numbers which is the completion of Q under the ordinary absolute value. The analogue of the field of complex numbers is obtained by taking the algebraic closure of Q_p and then completing this field under the extension of the p-adic valuation. We denote this field by Ω_p. It is now possible to consider analytic functions over Ω_p. If a function $f(z)$ is given by a power series with coefficients which are in the complex field and Ω_p, that is, if $f^{(n)}(0)$ is in both fields for $n = 0, 1, \cdots$, then $f(z)$ may be considered to be simultaneously a function of a complex and a p-adic variable. For

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instance the coefficients could be rational. In fact, it is meaningful to speak of an algebraic number as belonging to both Ω_p and the complex field. Indeed let α be algebraic. The minimal equation for α may factor over Q_p. Let $a_0x^n + \cdots + a_n$ be an irreducible factor over Q_p of the minimal equation where $a_0, \cdots, a_n$ are relatively prime integers in Q_p. Then the equation $a_0x^n + \cdots + a_n = 0$ has a solution in Ω_p. One of these solutions is identified with α and the extension of the p-adic valuation on Q gives

$$|\alpha|_p = |a_n/a_0|_p^{1/n}.$$

Let $|a|_\infty$ denote the ordinary absolute value of the number a.

Let

$$f_\mu(z) = \sum_{i=0}^{\infty} a_{i\mu} z^i \qquad (\mu = 1, \cdots, m) \tag{1}$$

be m functions of a complex and p-adic variable z where each $a_{i\mu}$ is algebraic. Let $\{z_\nu\}$ be a sequence of distinct algebraic points. Assume that

$$\lim_{\nu\to\infty} |z_\nu|_V = 0$$

for $V = \infty, p$. For example we could have $z_\nu = \nu!/p_{\nu!}$ where p_i denotes the ith prime. Suppose that $f_\mu(z_\nu)$ is the same algebraic number regardless of which field z_ν is in. For some index ν_0 let

$$P_\nu(V) = |z_{\nu_0+\nu+1} - z_{\nu_0+1}|_V \cdots |z_{\nu_0+\nu+1} - z_{\nu_0+\nu}|_V$$

and

$$\rho(V) = \liminf_{\nu\to\infty} \frac{\log|\log P_\nu(V)|_\infty}{\log \nu}.$$

Then $\rho(V)$ satisfies $1 \leqq \rho(V) \leqq \infty$ and is a measure of the rate of convergence of the sequence $\{z_\nu\}$; the more rapidly the sequence converges, the larger $\rho(V)$. Let

$$\gamma_\mu = \limsup_{\nu\to\infty} \frac{\log\log 3H(f_\mu(z_\nu))}{\log \nu}, \qquad \prod_{\mu=1}^{m} \deg f_\mu(z_\nu) \leqq h_\nu$$

with h_ν nondecreasing. Let

$$D_\nu = \sum_{\chi=\nu_0+1}^{\nu_0+\nu} h_\chi, \qquad \deg Q(f_1(z_{\nu_0+\nu}), \cdots, f_m(z_{\nu_0+\nu})) \leqq d_\nu$$

and

$$\alpha = \limsup_{\nu\to\infty} \frac{\log(D_\nu^{1/m} d_{\nu+1})}{\log \nu}\,.$$

THEOREM 1. *Under the above conditions the functions $f_1(z), \cdots, f_m(z)$ are algebraically dependent if they have a positive complex and p-adic radius of convergence and if*

$$\alpha + \frac{1}{m}\sum_{\mu=1}^{m} \gamma_\mu < \max_{V=\infty,p} \rho(V). \tag{2}$$

REMARK. Two sequences $z_\nu(\infty)$, $z_\nu(p)$ of possibly transcendental complex and p-adic numbers may be used. In which case α, $f_\mu(z_\nu(V))$ and γ_μ would depend on V, condition (2) is replaced by

$$\max_{V=\infty,p}\left(\alpha(V) + \frac{1}{m}\sum_{\mu=1}^{m}\gamma_\mu(V)\right) < \max_{V=\infty,p} \rho(V),$$

and h_χ is replaced by $\max_V h_\chi(V)$.

PROOF. The proof is somewhat similar to arguments in the paper [2] so we shall only outline the details.

Let

$$\phi(z) = \sum_{\tau_\mu=0}^{t_\mu} c_{\tau_1\cdots\tau_m} f_1^{\tau_1}(z)\cdots f_m^{\tau_m}(z), \qquad (\mu = 1, \cdots, m). \tag{3}$$

We shall show that for suitably chosen integers t_μ there exists integers $c_{\tau_1\cdots\tau_m}$ not all zero and not too large so that $\phi(z)\equiv 0$. Consider the n equations

$$\phi(z) = 0, \qquad z = z_{\nu_0+1}, \cdots, z_{\nu_0+n}, \tag{4}$$

in $c_{\tau_1\cdots\tau_m}$. Let $\beta_\mu = \beta_\mu(\nu) = f_\mu(z_\nu)$, $\delta(\mu) = \delta(\mu,\nu) = \deg \beta_\mu$ and $b_{\delta(\mu)} = b_{\delta(\mu)}(\nu)$ be the denominator of β_μ for $\mu = 1, \cdots, m$. Then the coefficients in (4) can be expressed as

$$\begin{aligned}\beta_1^{\tau_1}\cdots\beta_m^{\tau_m} &= b_{\delta(1)}^{-\tau_1}\cdots b_{\delta(m)}^{-\tau_m}\\ &\quad\cdot \sum_{\sigma_\mu=0}^{\delta(\mu)-1} a(\tau_1,\cdots,\tau_m,\sigma_1,\cdots,\sigma_m)\beta_1^{\sigma_1}\cdots\beta_m^{\sigma_m}\end{aligned} \tag{5}$$

where the a's are integers which satisfy

$$|\, a(\tau_1,\cdots,\tau_m,\sigma_1,\cdots,\sigma_m)\,|_\infty \leqq \prod_{\mu=1}^{m} (2H(\beta_\mu))^{\tau_\mu}.$$

If we substitute (5) into (4) and set equal to zero the resulting coefficients of $\beta_1^{\tau_1}\cdots\beta_m^{\tau_m}$ we obtain

$$(6)\qquad \sum_{\tau_\mu=0}^{t_\mu} b_{\delta(1)}^{t_1-\tau_1}\cdots b_{\delta(m)}^{t_m-\tau_m}\, a(\tau_1,\cdots,\tau_m,\sigma_1,\cdots,\sigma_m)c_{\tau_1\cdots\tau_m}=0$$

for $\nu=\nu_0+1,\cdots,\nu_0+n;\ \sigma_\mu=0,\cdots,\delta(\mu)-1;\ \mu=1,\cdots,m$. Thus we have a system (6) of at most D_n equations with integral coefficients in the $\prod_{\mu=1}^m(t_\mu+1)$ unknowns $c_{\tau_1\cdots\tau_m}$. Let

$$t_\mu=\left[(2D_n)^{1/m}(\nu_0+n)^{(1/m)\sum_{k=1}^m\gamma_k-\gamma_\mu}\right]$$

where $[x]$ denotes the greatest integer in x. Then the number of unknowns is at least twice the number of equations. The coefficients in (6) have, for a given $\epsilon>0$, absolute value at most

$$\prod_{\mu=1}^m\left\{2H(f_\mu(z_\nu))\right\}^{t_\mu}\leqq\exp(c_1D_n^{1/m}n^{(1/m)\sum_{k=1}^m\gamma_k+\epsilon})$$

where c_1 and the c_i's to follow are constants. Thus (6) has a solution in integers $c_{\tau_1\cdots\tau_m}$, not all zero, for which

$$\left|c_{\tau_1\cdots\tau_m}\right|_\infty\leqq\exp(c_2D_n^{1/m}n^{(1/m)\sum_{k=1}^m\gamma_k+\epsilon}).$$

Hence (4) holds. We shall now show that if $\phi(z)=0$ for $z=z_{\nu_0+1},\cdots,z_{\nu_0+n_1}$ then $\phi(z)=0$ for $z=z_{\nu_0+n_1+1}$ if $n_1\geqq n$ and n is sufficiently large. By applying the maximum principle to the analytic function

$$\frac{\phi(z)}{(z-z_{\nu_0+1})\cdots(z-z_{\nu_0+n_1})}$$

in some small closed disk about the origin we obtain an upper bound for $\left|\phi(z_{\nu_0+n_1+1})\right|_V$.

An upper bound for the conjugates can be obtained directly from (3). Let K be the denominator of $\phi(z_{\nu_0+n_1+1})$. Then if we combine these estimates we conclude that the norm of the algebraic integer $K\phi(z_{\nu_0+n_1+1})$ satisfies

$$(7)\qquad \begin{aligned}&\prod_{V=\infty,p}\left|N(K\phi(z_{\nu_0+n_1+1}))\right|_V\\ &\leqq\exp\left(c_3n_1^{\alpha+(1/m)\sum_{\mu=1}^m\gamma_\mu+2\epsilon}+c_4n_1-\sum_V n_1^{\rho(V)-\epsilon/2}\log n_1\right).\end{aligned}$$

If $\rho(V)\neq1$ for some V we see by (2) that the right side of (7) is less than 1. Thus $\phi(z_{\nu_0+n_1+1})=0$ for otherwise the left side of (7) would be at least 1. Thus $\phi(z)$ vanishes at the entire sequence $z_{\nu_0+1},z_{\nu_0+2},\cdots$ and hence it must be identically zero. If $\rho(V)=1$ for each V, then the term

$$\sum_V n_1^{\rho(V)-\epsilon/2} \log n_1$$

may be replaced by $c_5 n_1$ with $c_5 > c_4$ and again $\phi(z) \equiv 0$.

If equality holds in (2), then more complicated conditions are required. Here it becomes significant to consider, say, r primes $p_1, \cdots, p_r$. Let $f_\mu(z)$, $\mu = 1, \cdots, m$, be given by (1) with $a_{i\mu}$ algebraic. Let $z_\mu(V)$ be a sequence of distinct points, not necessarily algebraic, for $V = \infty, p_1, \cdots, p_r$ with $z_\nu(\infty)$ complex and $z_\nu(p_i)$ in Ω_{p_i}, $i = 1, \cdots, r$. Suppose that $\lim_{\nu\to\infty} |z_\nu(V)|_V = 0$ for $V = \infty, p_1, \cdots, p_r$ and that $f_\mu(z_\nu(V))$ is algebraic for $\mu = 1, \cdots, m$; $\nu = 1, 2, \cdots$; $V = \infty, p_1, \cdots, p_r$. For simplicity in the statement of the theorem, we assume that the degree of $f_\mu(z_\nu(V))$ is bounded. Here $f_\mu(z_\nu(V))$ and γ_μ depend on V. Let

$$\Omega_\mu(V) = \limsup_{\nu\to\infty} \frac{\log H(f_\mu(z_\nu(V)))}{\nu^{\gamma_\mu(V)} \log \nu}$$

and

$$\sigma(V) = \liminf_{\nu\to\infty} \frac{|\log P_\nu(V)|_\infty}{\nu^{\rho(V)} \log \nu}.$$

Then $\sigma(V)$ is a refinement of the measure $\rho(V)$ of the sequence $\{z_\nu(V)\}$. Let

$$\deg Q(f_1(z_\nu(V)), \cdots, f_m(z_\nu(V))) \leqq d$$

for $\nu = 1, 2, \cdots$ and $V = \infty, p_1, \cdots, p_r$.

THEOREM 2. *Under the above conditions the functions* $f_1(z), \cdots, f_m(z)$ *are algebraically dependent if they have a positive complex and* p_i*-adic radius of convergence for* $i = 1, \cdots, r$ *and if either* (a) *or* (b) *below hold.*

(a) $\max_V (1/m + (1/m) \sum_{\mu=1}^m \gamma_\mu(V)) < \max_V \rho(V)$.

(b) $1/m + (1/m) \sum_{\mu=1}^m \gamma_\mu(V) = \rho(V)$ *for each* V, $\rho(\infty) = \rho(p_1) = \cdots = \rho(p_r) \geqq \gamma_\mu$, *for each* μ, $\Omega_\mu(V) < \infty$, *for each* V *and* μ *and*

$$4md^2 \sum_V \left(2(r+1) \prod_{\mu=1}^m \Omega_\mu(V)\right)^{1/m} < \sum_V \sigma(V).$$

The proof is similar to that of Theorem 1.

REFERENCES

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