ON THE CONSTRUCTION OF UPPER RADICAL PROPERTIES

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Kurosh has proved the following theorem [**1**], [**2**].

THEOREM 1 (KUROSH). *Let $\mathfrak{M}$ be a class of rings with the property that if $R \in \mathfrak{M}$ and I is a nonzero ideal of R, then I can be mapped homomorphically onto a nonzero ring in $\mathfrak{M}$. Then there exists an upper radical property determined by $\mathfrak{M}$.*

For definitions of terms used in this paper, see Divinsky [**2**].

However, it was unknown whether every class of rings determines an upper radical property. The purpose of this paper is to solve this problem. The construction is similar to those in [**3**], [**4**].

We will need the following theorem.

THEOREM 2 (ANDERSON, DIVINSKY AND SULINSKI [**2**], [**5**]). *If $\mathfrak{S}$ is a radical property, then for any ring R and any ideal I of R, the $\mathfrak{S}$-radical of I is an ideal of R.*

THEOREM 3. *Every class of rings $\mathfrak{M}$ determines an upper radical property.*

PROOF. For each ring R, let $D_1(R) = \{I: I \text{ is an ideal of } R\}$. Assuming $D_n(R)$ has been defined, let $D_{n+1}(R) = \{I: I \text{ is an ideal of some ring in } D_n(R)\}$ and defined $D(R) = \bigcup\{D_n(R): n = 1, 2, \cdots\}$. Let $\overline{\mathfrak{M}}$ be the class of all rings A such that A is isomorphic to some ring in $D(R)$ for some R in $\mathfrak{M}$. If A is an element of $\overline{\mathfrak{M}}$, then there exists a positive integer n and a ring R an element of $\mathfrak{M}$ such that A is isomorphic to a ring I in $D_n(R)$. Therefore, if J is a nonzero ideal of A, then J is isomorphic and hence homomorphic to an ideal J' of I and $J' \in D_{n+1}(R)$ and hence $J' \in \overline{\mathfrak{M}}$ and $\overline{\mathfrak{M}}$ satisfies the hypothesis of Theorem 1. Thus $\overline{\mathfrak{M}}$ determines an upper radical property $\mathfrak{S}_{\overline{\mathfrak{M}}}$ and every ring in $\overline{\mathfrak{M}}$ and in particular in $\mathfrak{M}$ is $\mathfrak{S}_{\overline{\mathfrak{M}}}$-semisimple.

Next we want to show that $\mathfrak{S}_{\overline{\mathfrak{M}}}$ is the upper radical property determined by $\mathfrak{M}$. Let $\mathfrak{S}$ be any radical property such that every ring in $\mathfrak{M}$ is $\mathfrak{S}$-semisimple. For each $R \in \mathfrak{M}$, by Theorem 2, every ring in $D_1(R)$ is $\mathfrak{S}$-semisimple and by induction, every ring in $D(R)$ is $\mathfrak{S}$-semisimple and hence every ring in $\overline{\mathfrak{M}}$ is $\mathfrak{S}$-semisimple. Since $\mathfrak{S}_{\overline{\mathfrak{M}}}$ is the upper radical property determined by $\overline{\mathfrak{M}}$, it follows that $\mathfrak{S} \subset \mathfrak{S}_{\overline{\mathfrak{M}}}$.

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BIBLIOGRAPHY

1. A. G. Kurosh, *Radicals of rings and algebras*, Mat. Sb. N.S. **37**(**75**) (1953), 13–26. (Russian)

2. N. J. Divinski, *Rings and radicals*, Univ. of Toronto Press, Toronto, 1965.

3. Y. L. Lee, *A characterization of Baer lower radical property*, Kyungpook Math. J. **7** (1967).

4. ———, *On the construction of lower radical properties*, Pacific J. Math (to appear).

5. R. Anderson, N. Divinsky and A. Sulinski, *Hereditary radicals in associative and alternative rings*, Canad. J. Math. **7** (1965), 594–603.

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