IRREDUCIBLE REPRESENTATIONS OF A SIMPLE LIE ALGEBRA ADMITTING A ONE-DIMENSIONAL WEIGHT SPACE

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Introduction. Let $\mathcal{L}$ be a finite-dimensional, simple Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic zero. In this paper we shall study the family of all linear irreducible (finite- or infinite-dimensional) representations of $\mathcal{L}$ which admit a one-dimensional weight space. It is well known that this family includes all finite-dimensional irreducible representations of $\mathcal{L}$. More generally, we know that the weight space corresponding to the dominant weight function of Harish-Chandra’s irreducible representations with dominant weight functions is one dimensional [5]. Finally in a forthcoming paper, Bouwer studies a class of linear irreducible representations, called standard representations, which do not possess a dominant weight function but do admit a “characteristic weight function” whose weight space is one dimensional. It is the purpose of this paper to provide a characterization of all linear irreducible representations of $\mathcal{L}$ admitting a one-dimensional weight space.

Preliminaries. Let $\Gamma$ denote the set of all roots of $\mathcal{L}$ with respect to the Cartan subalgebra $\mathfrak{K}$. Assuming that the roots of $\mathcal{L}$ have been given a fixed order, let $\Gamma^0$ and $\Gamma^+$ denote the simple and positive roots of $\mathcal{L}$ respectively. Finally, let

$$\Omega = \{X_\beta, Y_\beta, H_\alpha \mid \beta \in \Gamma^+; \alpha \in \Gamma^0\}$$

denote the usual Cartan basis of $\mathcal{L}$ where, for each simple root $\alpha \in \Gamma^0$, we have $[X_\alpha, Y_\alpha] = H_\alpha; [X_\alpha, H_\alpha] = 2X_\alpha; \text{and } [Y_\alpha, H_\alpha] = -2Y_\alpha$ where $[\cdot, \cdot]$ denotes the Lie product in $\mathcal{L}$.

By the Poincaré-Birkhoff-Witt Theorem, the universal enveloping algebra $U(\mathcal{L})$ of $\mathcal{L}$ admits a linear basis consisting of all elements of the form:

$$\prod_{\beta \in \Gamma^+} Y_\beta^{m(\beta)} \prod_{\alpha \in \Gamma^0} H_\alpha^{k(\alpha)} \prod_{\beta \in \Gamma^+} X_\beta^{n(\beta)},$$

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3 For elementary definitions and properties of Lie algebras see [1], [6], [7].

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where the exponents \( m(\beta), k(\alpha) \) and \( n(\beta) \) are all nonnegative integers and the products \( \prod \) each respect a fixed order over their index sets. Consider now the linear subspace \( C(\mathcal{L}) \) of \( U(\mathcal{L}) \) generated by all elements of the form (1) which also satisfy the following condition:

\[
\sum_{\beta \in \Gamma^+} (n(\beta) - m(\beta)) \beta = 0.
\]

By a simple (but tedious) inductive proof, we can show that \( C(\mathcal{L}) \) is in fact a subalgebra of \( U(\mathcal{L}) \), which we shall call the cycle subalgebra of \( U(\mathcal{L}) \). An element of the form (1) which also satisfies condition (2) is called an elementary cycle if, when considered as a commutative monomial, it cannot be written as the product of two nontrivial elements of \( C(\mathcal{L}) \). (For example \( Y_\beta X_\beta \) is an elementary cycle, whereas \( Y_\beta H_\alpha X_\delta \) is not an elementary cycle.) Using induction on the number of factors in the basis elements of \( C(\mathcal{L}) \) and the commutant relations on the basis elements of \( \mathcal{L} \), it is easily shown that the elementary cycles of \( C(\mathcal{L}) \) form a system of (algebra) generators of \( C(\mathcal{L}) \).

The cycle subalgebra \( C(\mathcal{L}) \) was originally introduced by Chevalley in his algebraic proof of the existence of simple Lie algebras corresponding to the various Cartan matrices \([4]\). Its relevance to our present study is brought out by the following result.

**Lemma 1.** Let \( (\rho, V) \) be an irreducible representation of \( U(\mathcal{L}) \) which admits \( \lambda \) as a weight function. Then, if \( v \) is any nonzero element in the \( \lambda \)-weight space \( V_\lambda \) of \( (\rho, V) \), we have

\[
\rho(C(\mathcal{L}))v = V_\lambda.
\]

**Proof.** This follows as a special case of Lemma 3 in Bouwer's paper \([3]\).

Let us now suppose the \( V_\lambda \) in the previous lemma is a one-dimensional weight space of the irreducible representation \( (\rho, V) \). Then it is obvious that the identity (3) naturally induces an algebra homomorphism \( \gamma: C(\mathcal{L}) \to \mathfrak{g} \), where, for each \( c \in C(\mathcal{L}) \), we define \( \gamma(c) = f \), where \( \rho(c)v = fv \). It is readily verified that the map \( \gamma \) so defined is an algebra homomorphism and is independent of the particular choice of \( v \in V \). Moreover, \( \gamma \) when restricted to the Cartan subalgebra, \( \mathfrak{h} \) coincides with the weight function \( \lambda \). In general, any algebra homomorphism \( \zeta: C(\mathcal{L}) \to \mathfrak{g} \) for which there exists a nonzero vector \( v \in V \) with \( \rho(c)v = \zeta(c)v \) for all \( c \in C(\mathcal{L}) \) will be called a mass function of the representation \( (\rho, V) \).

**Construction of representations.** So far we have established that every irreducible representation \( (\rho, V) \) which admits a one-dimen-
sional weight function $\lambda$ also admits a mass function $\gamma$ which is an extension of $\lambda$. We shall now show that this process is essentially reversible—that is, given a nonzero algebra homomorphism $\gamma: C(\mathfrak{g}) \rightarrow \mathbb{F}$, we shall construct a unique irreducible representation which admits $\gamma$ as a mass function. Moreover this representation will admit $\gamma \downarrow \mathfrak{h}$ as a one-dimensional weight function.

Let $\gamma: C(\mathfrak{g}) \rightarrow \mathbb{F}$ be a fixed algebra homomorphism. We first note that a linear basis of $U(\mathfrak{g})$, called the $\gamma$-basis, is provided by the set of all elements of the form

$$ x(u) \prod (c - \gamma(c) \cdot 1)^{n(c)} $$

where the indices $n(c)$ are all nonnegative integers, $\prod$ is an ordered product ranging over the elementary cycles (different from 1), the elements $x(u)$ are well determined elements of the form (1) which do not contain any cycle different from 1 and priority of construction is given to cycles which appear to the right in the above product. We also denote by $I^*(\gamma)$ the left ideal of $U(\mathfrak{g})$ generated by $\{ c - \gamma(c) \cdot 1 \mid c \text{ is an elementary cycle of } U(\mathfrak{g}) \}$.

**Lemma 2.** $I^*(\gamma) \neq U(\mathfrak{g})$.

**Proof.** Clearly, $I^*(\gamma) \neq U(\mathfrak{g})$ iff $1 \notin I^*(\gamma)$. To establish that $1 \notin I^*(\gamma)$, it suffices to show that for any $\gamma$-basis element $u$ of $U(\mathfrak{g})$ and for any elementary cycle $c$ of $C(\mathfrak{g})$, the element $u(c - \gamma(c) \cdot 1)$ contains no nonzero multiples of 1 when expressed in terms of the $\gamma$-basis. By the form of the $\gamma$-basis elements, this may be reduced to showing that $(c_1 - \gamma(c_1) \cdot 1)(c_2 - \gamma(c_2) \cdot 1)$ when expressed in terms of the $\gamma$-basis of $U(\mathfrak{g})$ contains no nonzero multiple of 1, for any two elementary cycles $c_1$ and $c_2$. This follows from the fact that $\gamma$ was assumed to be an algebra homomorphism.

**Lemma 3.** There exists a unique maximal left ideal $I(\gamma)$ of $U(\mathfrak{g})$ which contains $I^*(\gamma)$.

**Proof.** The existence of at least one maximal left ideal containing $I^*(\gamma)$ follows from an elementary argument using Zorn's Lemma. In order to establish the uniqueness of this maximal left ideal we note that there is a one-to-one correspondence between left ideals containing $I^*(\gamma)$ and invariant subspaces of the left regular representation $\tilde{\rho}$ of $U(\mathfrak{g})$ modulo $I^*(\gamma)$. Let $V'$ be an invariant subspace of $(\tilde{\rho}, U(\mathfrak{g})/I^*(\gamma))$. Since $\gamma$ restricted to $\mathfrak{h}$ is a one-dimensional weight function of $\tilde{\rho}$ it follows that if $V'$ is a proper invariant subspace then $V' \bigoplus_{\lambda \neq \gamma \downarrow \mathfrak{h}} V_{\lambda}$ where $V_{\lambda}$ ranges over all $\lambda$-weight spaces of $\tilde{\rho}$ different from $V_{\gamma \downarrow \mathfrak{h}}$. Therefore the sum of all proper $\tilde{\rho}$-invariant sub-
spaces is again a proper $\bar{p}$-invariant subspace and is uniquely determined as the largest proper $\bar{p}$-invariant subspace.

**Lemma 4.** The left regular representation of $U(\mathfrak{L})$ modulo $I(\gamma)$ is a linear irreducible representation of $U(\mathfrak{L})$ admitting $\gamma$ as a mass function.

**Proof.** The maximality of $I(\gamma)$ insures that this representation is irreducible. In order to check that $\gamma$ is a mass function, we note that $1 + I(\gamma) \neq I(\gamma)$, and, for any element $c \in C(\mathfrak{L})$, we have

$$c(1 + I(\gamma)) = c + I(\gamma),$$

$$\equiv \gamma(c) \cdot 1 + I(\gamma) \mod I(\gamma),$$

$$\equiv \gamma(c)(1 + I(\gamma)) \mod I(\gamma).$$

**Lemma 5.** Let $(\rho, V)$ be an irreducible representation of $U(\mathfrak{L})$ admitting $\gamma$ as a mass function. Then $(\rho, V)$ is equivalent to the left regular representation of $U(\mathfrak{L})$ modulo $I(\gamma)$.

**Proof.** Let $v$ be a fixed nonzero element of the $(\gamma \downarrow \mathfrak{C})$-weight space of $(\rho, V)$. Then the map $\phi: U(\mathfrak{L})/I(\gamma) \to V$ defined by $\phi: u + I(\gamma) \to \rho(u)v$ is an isomorphism which establishes the equivalence of the two representations.

Summarizing the results of Lemmas 2 through 5 we have the following theorem.

**Theorem.** Given any algebra homomorphism $\gamma: C(\mathfrak{L}) \to \mathfrak{F}$ then there exists a unique irreducible representation of $U(\mathfrak{L})$ admitting $\gamma$ as a mass function. Moreover, the restriction of $\gamma$ to the Cartan subalgebra $\mathfrak{C}$ is a one-dimensional weight function of this representation.

**Remark.** A more detailed study of the properties of irreducible representations characterized by mass functions $w$ will be published at a later date.

**Bibliography**