IRREDUCIBLE REPRESENTATIONS OF A SIMPLE LIE ALGEBRA ADMITTING A ONE-DIMENSIONAL WEIGHT SPACE[1]

F. W. LEMIRE[2]

Introduction.[3] Let $\mathcal{L}$ be a finite-dimensional, simple Lie algebra over an algebraically closed field $\mathfrak{F}$ of characteristic zero. In this paper we shall study the family of all linear irreducible (finite- or infinite-dimensional) representations of $\mathcal{L}$ which admit a one-dimensional weight space. It is well known that this family includes all finite-dimensional irreducible representations of $\mathcal{L}$. More generally, we know that the weight space corresponding to the dominant weight function of Harish-Chandra's irreducible representations with dominant weight functions is one dimensional [5]. Finally in a forthcoming paper, Bouwer studies a class of linear irreducible representations, called standard representations, which do not possess a dominant weight function but do admit a "characteristic weight function" whose weight space is one dimensional. It is the purpose of this paper to provide a characterization of all linear irreducible representations of $\mathcal{L}$ admitting a one-dimensional weight space.

Preliminaries. Let Γ denote the set of all roots of $\mathcal{L}$ with respect to the Cartan subalgebra $\mathcal{H}$. Assuming that the roots of $\mathcal{L}$ have been given a fixed order, let Γ^0 and Γ^+ denote the simple and positive roots of $\mathcal{L}$ respectively. Finally, let

$$\Omega = \{X_\beta, Y_\beta, H_\alpha \mid \beta \in \Gamma^+; \alpha \in \Gamma^0\}$$

denote the usual Cartan basis of $\mathcal{L}$ where, for each simple root $\alpha \in \Gamma^0$, we have $[X_\alpha, Y_\alpha] = H_\alpha$; $[X_\alpha, H_\alpha] = 2X_\alpha$; and $[Y_\alpha, H_\alpha] = -2Y_\alpha$ where $[\cdot\ ,\ \cdot]$ denotes the Lie product in $\mathcal{L}$.

By the Poincaré-Birkhoff-Witt Theorem, the universal enveloping algebra $U(\mathcal{L})$ of $\mathcal{L}$ admits a linear basis consisting of all elements of the form:

$$\prod_{\beta\in\Gamma^+} Y_\beta^{m(\beta)} \prod_{\alpha\in\Gamma^0} H_\alpha^{k(\alpha)} \prod_{\beta\in\Gamma^+} X_\beta^{n(\beta)}, \tag{1}$$

Received by the editors May 1, 1967.

[1] The results of this paper constitute one part of a doctoral dissertation written at Queen's University, Kingston under the supervision of Professor A. J. Coleman.

[2] The author is indebted to the Woodrow Wilson Fellowship and the National Research Council for their financial assistance during the period of this research.

[3] For elementary definitions and properties of Lie algebras see [1], [6], [7].

where the exponents $m(\beta)$, $k(\alpha)$ and $n(\beta)$ are all nonnegative integers and the products $\prod$ each respect a fixed order over their index sets. Consider now the linear subspace $C(\mathfrak{L})$ of $U(\mathfrak{L})$ generated by all elements of the form (1) which also satisfy the following condition:

$$\sum_{\beta\in\Gamma^+} (n(\beta) - m(\beta))\beta = 0. \tag{2}$$

By a simple (but tedious) inductive proof, we can show that $C(\mathfrak{L})$ is in fact a subalgebra of $U(\mathfrak{L})$, which we shall call the *cycle subalgebra* of $U(\mathfrak{L})$. An element of the form (1) which also satisfies condition (2) is called an *elementary cycle* if, when considered as a commutative monomial, it cannot be written as the product of two nontrivial elements of $C(\mathfrak{L})$. (For example $Y_\beta X_\beta$ is an elementary cycle, whereas $Y_\beta H_\alpha X_\beta$ is not an elementary cycle.) Using induction on the number of factors in the basis elements of $C(\mathfrak{L})$ and the commutant relations on the basis elements of $\mathfrak{L}$, it is easily shown that the elementary cycles of $C(\mathfrak{L})$ form a system of (algebra) generators of $C(\mathfrak{L})$.

The cycle subalgebra $C(\mathfrak{L})$ was originally introduced by Chevalley in his algebraic proof of the existence of simple Lie algebras corresponding to the various Cartan matrices [4]. Its relevance to our present study is brought out by the following result.

LEMMA 1. *Let (ρ, V) be an irreducible representation of $U(\mathfrak{L})$ which admits λ as a weight function. Then, if v is any nonzero element in the λ-weight space V_λ of (ρ, V), we have*

$$\rho(C(\mathfrak{L}))v = V_\lambda. \tag{3}$$

PROOF. This follows as a special case of Lemma 3 in Bouwer's paper [3].

Let us now suppose the V_λ in the previous lemma is a one-dimensional weight space of the irreducible representation (ρ, V). Then it is obvious that the identity (3) naturally induces an algebra homomorphism γ: $C(\mathfrak{L})\to\mathfrak{F}$, where, for each $c\in C(\mathfrak{L})$, we define $\gamma(c)=f$, where $\rho(c)v=fv$. It is readily verified that the map γ so defined is an algebra homomorphism and is independent of the particular choice of $v\in V$. Moreover, γ when restricted to the Cartan subalgebra, $\mathfrak{H}$ coincides with the weight function λ. In general, any algebra homomorphism ζ: $C(\mathfrak{L})\to\mathfrak{F}$ for which there exists a nonzero vector $v\in V$ with $\rho(c)v=\zeta(c)v$ for all $c\in C(\mathfrak{L})$ will be called a mass function of the representation (ρ, V).

Construction of representations. So far we have established that every irreducible representation (ρ, V) which admits a one-dimensional weight function λ also admits a mass function γ which is an extension of λ. We shall now show that this process is essentially reversible—that is, given a nonzero algebra homomorphism $\gamma: C(\mathfrak{L}) \to \mathfrak{F}$, we shall construct a unique irreducible representation which admits γ as a mass function. Moreover this representation will admit $\gamma \downarrow \mathfrak{H}$ as a one-dimensional weight function.

Let $\gamma: C(\mathfrak{L}) \to \mathfrak{F}$ be a fixed algebra homomorphism. We first note that a linear basis of $U(\mathfrak{L})$, called the *γ-basis*, is provided by the set of all elements of the form

$$x(u)\left[\prod (c - \gamma(c)\cdot 1)^{n(c)}\right] \tag{4}$$

where the indices $n(c)$ are all nonnegative integers, $\prod$ is an ordered product ranging over the elementary cycles (different from 1), the elements $x(u)$ are well determined elements of the form (1) which do not contain any cycle different from 1 and priority of construction is given to cycles which appear to the right in the above product. We also denote by $I^*(\gamma)$ the left ideal of $U(\mathfrak{L})$ generated by $\{c-\gamma(c)\cdot 1 \mid c$ is an elementary cycle of $U(\mathfrak{L})\}$.

LEMMA 2. $I^*(\gamma) \neq U(\mathfrak{L})$.

PROOF. Clearly, $I^*(\gamma) \neq U(\mathfrak{L})$ iff $1 \notin I^*(\gamma)$. To establish that $1 \notin I^*(\gamma)$, it suffices to show that for any γ-basis element u of $U(\mathfrak{L})$ and for any elementary cycle c of $C(\mathfrak{L})$, the element $u(c-\gamma(c)\cdot 1)$ contains no nonzero multiples of 1 when expressed in terms of the γ-basis. By the form of the γ-basis elements, this may be reduced to showing that $(c_1-\gamma(c_1)\cdot 1)(c_2-\gamma(c_2)\cdot 1)$ when expressed in terms of the γ-basis of $U(\mathfrak{L})$ contains no nonzero multiple of 1, for any two elementary cycles c_1 and c_2. This follows from the fact that γ was assumed to be an algebra homomorphism.

LEMMA 3. *There exists a unique maximal left ideal $I(\gamma)$ of $U(\mathfrak{L})$ which contains $I^*(\gamma)$.*

PROOF. The existence of at least one maximal left ideal containing $I^*(\gamma)$ follows from an elementary argument using Zorn's Lemma. In order to establish the uniqueness of this maximal left ideal we note that there is a one-to-one correspondence between left ideals containing $I^*(\gamma)$ and invariant subspaces of the left regular representation $\tilde{\rho}$ of $U(\mathfrak{L})$ modulo $I^*(\gamma)$. Let V' be an invariant subspace of $(\tilde{\rho}, U(\mathfrak{L})/I^*(\gamma))$. Since γ restricted to $\mathfrak{H}$ is a one-dimensional weight function of $\tilde{\rho}$ it follows that if V' is a proper invariant subspace then $V' \sum\subseteq_{\lambda\neq\gamma\downarrow\mathfrak{H}} V_\lambda$ where V_λ ranges over all λ-weight spaces of $\tilde{\rho}$ different from $V_{\gamma\downarrow\mathfrak{H}}$. Therefore the sum of all proper $\tilde{\rho}$-invariant subspaces is again a proper $\tilde{\rho}$-invariant subspace and is uniquely determined as the largest proper $\tilde{\rho}$-invariant subspace.

LEMMA 4. *The left regular representation of $U(\mathfrak{L})$ modulo $I(\gamma)$ is a linear irreducible representation of $U(\mathfrak{L})$ admitting γ as a mass function.*

PROOF. The maximality of $I(\gamma)$ insures that this representation is irreducible. In order to check that γ is a mass function, we note that $1+I(\gamma)\neq I(\gamma)$, and, for any element $c\in C(\mathfrak{L})$, we have

$$\begin{aligned} c(1 + I(\gamma)) &= c + I(\gamma), \\ &\equiv \gamma(c)\cdot 1 + I(\gamma) \mod I(\gamma), \\ &\equiv \gamma(c)(1 + I(\gamma)) \mod I(\gamma). \end{aligned}$$

LEMMA 5. *Let (ρ, V) be an irreducible representation of $U(\mathfrak{L})$admitting γ as a mass function. Then (ρ, V) is equivalent to the left regular representation of $U(\mathfrak{L})$ modulo $I(\gamma)$.*

PROOF. Let v be a fixed nonzero element of the $(\gamma \downarrow \mathfrak{H})$-weight space of (ρ, V). Then the map ϕ: $U(\mathfrak{L})/I(\gamma)\rightarrow V$ defined by ϕ: $u+I(\gamma)\rightarrow\rho(u)v$ is an isomorphism which establishes the equivalence of the two representations.

Summarizing the results of Lemmas 2 through 5 we have the following theorem.

THEOREM. *Given any algebra homomorphism γ: $C(\mathfrak{L})\rightarrow\mathfrak{F}$ then there exists a unique irreducible representation of $U(\mathfrak{L})$ admitting γ as a mass function. Moreover, the restriction of γ to the Cartan subalgebra $\mathfrak{H}$ is a one-dimensional weight function of this representation.*

REMARK. A more detailed study of the properties of irreducible representations characterized by mass functions w will be published at a later date.

BIBLIOGRAPHY

1. N. Bourbaki, *Groupes et algèbres de Lie*, Hermann, Paris, 1960.

2. I. Z. Bouwer, *Standard representations of simple Lie algebras*, Thesis, Univ. of Toronto, Toronto, 1964.

3. ———, *Standard representations of simple Lie algebras*, Canad. J. Math. (to appear)

4. C. Chevalley, *Sur la classification des algèbres de Lie simples et de leurs représentations*, C.R. Acad. Sci. Paris **227** (1948), 1136–1138.

5. Harish-Chandra, *On some applications of the universal enveloping algebra of a semi-simple Lie algebra*, Trans. Amer. Math. Soc. **70** (1951), 28–99.

6. N. Jacobson, *Lie algebras*, Interscience, New York, 1962.

7. Séminaire "Sophus Lie," Ecole Normale Supérieure, 1954/1955: *Théorie des algèbres de Lie. Topologie des groupes de Lie*, Secrétariat mathématique, Paris, 1955.

QUEEN'S UNIVERSITY, KINGSTON, ONTARIO