ON $\mathfrak{F}$-NORMALIZERS AND $\mathfrak{F}$-COVERING SUBGROUPS

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Recently, Carter and Hawkes [2] have generalized the construction of system normalizers of finite solvable groups, introducing the concept of an $\mathfrak{F}$-normalizer, where $\mathfrak{F}$ is any local formation. In this note we give a different proof of one of their main results, namely, each $\mathfrak{F}$-normalizer is contained in an $\mathfrak{F}$-covering subgroup. Our proof avoids the study of the imbedding of $\mathfrak{F}$-normalizers in maximal subgroups, and is similar to Huppert's proof that each system normalizer is contained in a Carter subgroup [**4**, p. 37].

We first recall the definitions. All groups in this note are solvable and finite. A *formation* of groups is a class of groups closed under homomorphic images and subdirect products. If formations $\mathfrak{F}(p)$, one for each prime p, are given, the *local formation* $\mathfrak{F}$ *locally defined* by $\{\mathfrak{F}(p)\}$ is the class of all groups G such that, whenever M is a chief factor of G, of order p^n, say, then the automorphism group induced on M by G belongs to $\mathfrak{F}(p)$. (We are assuming that $\mathfrak{F}(p)\neq\emptyset$ for each p.)

Let $\mathfrak{F}$ be locally defined by $\{\mathfrak{F}(p)\}$. For each p, let N_p be the unique minimal normal subgroup of G such that $G/N_p\in\mathfrak{F}(p)$. Let T^p be a p-complement of N_p. Then $D=\bigcap_p N(T^p)$ is an $\{\mathfrak{F}(p)\}$*-normalizer* of G. If $\mathfrak{F}(p)\subseteq\mathfrak{F}$, for each p, then the $\{\mathfrak{F}(p)\}$-normalizers depend only on $\mathfrak{F}$, and not on $\mathfrak{F}(p)$, and are called $\mathfrak{F}$*-normalizers*. An $\mathfrak{F}$*-covering subgroup* of G is a subgroup C such that $C\in\mathfrak{F}$ and, whenever $C\subseteq K\subseteq G$, $L\triangleleft K$ and $K/L\in\mathfrak{F}$, then $K=LC$. If $\mathfrak{F}$ is local, then $\mathfrak{F}$-covering subgroups of G exist, and are unique up to conjugacy [**3**].

We use below the (known) fact that a homomorphism maps $\mathfrak{F}$-normalizers onto $\mathfrak{F}$-normalizers, and $\mathfrak{F}$-covering subgroups onto $\mathfrak{F}$-covering subgroups.

THEOREM. *Let $\mathfrak{F}$ be locally defined by $\{\mathfrak{F}(p)\}$. If, for each p, either $\mathfrak{F}(p)\subseteq\mathfrak{F}$ or $\mathfrak{F}(p)$ is subgroup closed, then each $\mathfrak{F}$-covering subgroup of G contains an $\{\mathfrak{F}(p)\}$-normalizer of G.*

PROOF. Let C be an $\mathfrak{F}$-covering subgroup of G, and let N_p and T^p be defined as above. We say that the system $\{T^p\}$ *reduces* to C, if there exist p-complements of G, S^p, such that $T^p=S^p\cap N_p$, and $S^p\cap C$ is a p-complement of C. Given a system $\{C^p\}$ of p-complements of C, one can choose p-complements $\{S^p\}$ of G such that $C^p\subseteq S^p$. De-

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noting now $T^p = S^p \cap N_p$, the system $\{T^p\}$ reduces to C. Hence such systems exist.

We shall prove, by induction on $|G|$, that if $\{T^p\}$ reduces to C, then $D = \bigcap_p N(T^p)$ is contained in C.

If $G = C$, there is nothing to prove, so we assume $G \neq C$, so that $G \notin \mathfrak{F}$. Let M be a minimal normal subgroup of G. By induction, $DM/M \subseteq CM/M$, hence $D \subseteq CM$.

Suppose first that $CM \neq G$. Let $\{S^p\}$ be the p-complements of G defined above. Since $\{S^p\}$ reduces to C and $M \triangleleft G$, $\{S^p\}$ reduces to CM [1, Corollary 2.8]. Denote $U^p = S^p \cap CM$. Let Q_p be the minimal normal subgroup of CM, for which $CM/Q_p \in \mathfrak{F}(p)$. If $\mathfrak{F}(p) \subseteq \mathfrak{F}$, then the defining properties of C imply $G = CN_p$; therefore $CM/CM \cap N_p \cong G/N_p \in F(p)$. If $\mathfrak{F}(p)$ is subgroup closed, the same conclusion is true, since $CM/CM \cap N_p$ is isomorphic to a subgroup of G/N_p. Hence in any case $Q_p \subseteq CM \cap N_p$.

Now the system $\{T^p \cap Q_p\} = \{S^p \cap Q_p\} = \{U^p \cap Q_p\}$ reduces to C, which is an $\mathfrak{F}$-covering subgroup of CM. By induction, $\bigcap N_{CM}(T^p \cap Q_p)$ is contained in C. Obviously, $D \subseteq N(T^p \cap Q_p)$ for all p, so $D \subseteq C$.

Now assume $G = CM$. Let $|M| = q^n$, for some prime q. By assumption, $G \notin \mathfrak{F}$, but $G/M \cong C \in \mathfrak{F}$. Hence, by Gaschütz's construction of $\mathfrak{F}$-covering subgroups [3], $C = N(V^q)$, where V^q is a q-complement of $O_{q'}$ (G mod M). Since $|G: C| = q^n$, and $S^q \cap C$ is a q-complement of C, $S^q \subseteq C$, so also $T^q \subseteq C$. As $G/M \in \mathfrak{F}$, we must have $N_q \subseteq O_{q'q}$ (G mod M); therefore $T^q \subseteq O_{q'}$ (G mod M). As $T^q \subseteq C$, T^q normalizes V^q; therefore $T^q \subseteq V^q$, $T^q = V^q \cap N_q$. This implies $T^q \triangleleft C$. Since C is maximal, $N(T^q) = C$ or $N(T^q) = G$. In the second case N_q has a normal q-complement; hence $N_q \subseteq O_{q'q}(G)$, which is equivalent to G inducing on all chief factors of orders q^m a group belonging to $\mathfrak{F}(q)$. A chief factor of order p^m, $p \neq q$, is operator isomorphic to one of G/M; hence G certainly induces on it a group belonging to $\mathfrak{F}(p)$. Therefore $G \in \mathfrak{F}$, a contradiction. Hence $C = N(T^q)$ and $D \subseteq C$.

References

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