DIRECT DECOMPOSITION OF REGULAR SEMIGROUPS

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1. **Introduction.** If X is a semigroup, let E_X denote the set of idempotents of X. The major purpose of this paper is to show that S is a regular semigroup with $E_S \cong K \times B$, where K is a semilattice and B is a rectangular band, if and only if $S \cong T \times B$ where T is an inverse semigroup with $E_T \cong K$. In the case S is bisimple, the structure of S may be described completely mod groups for several classes of semilattices K (Remark 1). In the case S is a Clifford semigroup, its structure may also be completely determined (Remark 2). A characterization of certain classes of regular semigroups, including right groups, is an immediate consequence of the theorem.

We adopt the terminology and notation of [1].

2. **The decomposition theorem.**

THEOREM. *S is a regular semigroup such that $E_S \cong K \times B$ where K is a semilattice and B is a rectangular band if and only if $S \cong T \times B$ where T is an inverse semigroup with $E_T \cong K$. S is bisimple if and only if T is bisimple.*

PROOF. Let S be a regular semigroup such that $E_S \cong K \times B$ where K is a semilattice and B is a rectangular band. Let B be the set product $I \times J$ under the multiplication $(i, j)(k, s) = (i, s)$. If $a \in S$, $a \in R_{(e,i,j)} \cap L_{(f,k,s)}$ where (e, i, j), $(f, k, s) \in K \times B$ since S is a regular semigroup. Thus, $a = (e, i, j)a = a(f, k, s)$. Hence, $(g, r, t)a = (g, r, q)a$ and $a(g, p, q) = a(g, r, q)$ for $g \in K$, $r, p \in I$ and $t, q \in J$. (We will use this fact several times in the proof without further mention.) Therefore, $a \in (e, i, s) \cdot S(f, i, s)$ and

$$S = \bigcup \{(e, i, j)S(f, i, j) : e, f \in K, i \in I, j \in J\}.$$

Let i_0 and j_0 be fixed elements of I and J respectively. (The elements i_0 and j_0 may be selected arbitrarily. Then, they are fixed.)

We will define an isomorphism θ of

$$M = (\bigcup \{(e, i_0, j_0)S(f, i_0, j_0) : e, f \in K\}) \times B$$

onto $S = \bigcup \{(e, i, j)S(f, i, j) : e, f \in K, i \in I, j \in J\}$ in the following manner:

$$(x, i, j)\theta = (e, i, j)x(f, i, j) \quad \text{where} \quad (x, i, j) \in ((e, i_0, j_0)\, S(f, i_0, j_0)) \times B.$$

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First, we show that θ is single valued. Let $u=v=(x, i, j)\in(((e, i_0, j_0) S(f, i_0, j_0))\times B)\cap(((g, i_0, j_0)S(h, i_0, j_0))\times B)$.

Thus,

$$\begin{aligned}(e, i, j)x(f, i, j) &= (e, i, j)((g, i_0, j_0)x(h, i_0, j_0))(f, i, j)\\ &= (eg, i, j_0)x(hf, i_0, j)\\ &= (g, i, j)((e, i_0, j_0)x(f, i_0, j_0))(h, i, j)\\ &= (g, i, j)x(h, i, j).\end{aligned}$$

Hence $u\theta=v\theta$.

Next, we show that θ is one-to-one. Suppose that $u\theta=v\theta$ where $u\in(e, i_0, j_0)S(f, i_0, j_0)\times B$ and $v\in(g, i_0, j_0)S(h, i_0, j_0)\times B$. Let $u=(x, i, j)$ and $v=(y, k, s)$. Thus, $(e, i, j)x(f, i, j)=(g, k, s)y(h, k, s)=z$, say. Hence, $z=(e, i, j)z=(g, k, s)z$. Since S is a regular semigroup, there exists $z'\in S$ such that $zz'\in E_S$. Thus, suppose that $zz'=(t, p, q)$ where $t\in K$, $p\in I$, and $q\in J$. Hence, $(e, i, j)(t, p, q)=(g, k, s)(t, p, q)$, $(et, i, q)=(gt, k, q)$, and $i=k$. In a similar manner $j=s$. Thus,

$$(e, i, j)x(f, i, j) = (g, i, j)y(h, i, j) = (g, i, j_0)y(h, i_0, j).$$

Thus,

$$\begin{aligned}(g, i_0, j_0)(g, i, j_0)y(h, i_0, j)(h, i_0, j_0) &= (g, i_0, j_0)(e, i, j)x(f, i, j)(h, i_0, j_0),\\ (g, i_0, j_0)y(h, i_0, j_0) &= (ge, i_0, j)x(fh, i, j_0),\\ y &= (e, i_0, j_0)(g, i_0, j)x(h, i, j_0)(f, i_0, j_0).\end{aligned}$$

Hence, $(y, k, s)\in(e, i_0, j_0)S(f, i_0, j_0)\times B$. Thus

$$\begin{aligned}(e, i, j)x(f, i, j) &= (e, i, j)y(f, i, j),\\ (e, i, j)(x(f, i_0, j_0))(f, i, j)(f, i_0, j_0) &= (e, i, j)(y(f, i_0, j_0))(f, i, j)(f, i_0, j_0),\\ (e, i, j)x(f, i_0, j_0) &= (e, i, j)y(f, i_0, j_0),\\ (e, i, j)x &= (e, i, j)y,\\ (e, i_0, j_0)(e, i, j)((e, i_0, j_0)x) &= (e, i_0, j_0)(e, i, j)((e, i_0, j_0)y),\\ (e, i_0, j_0)x &= (e, i_0, j_0)y,\\ x &= y.\end{aligned}$$

Hence, $u=v$ and θ is one-to-one.

Next, we show that θ maps M onto S. Let $(e, i, j)x(f, i, j)\in S$. Thus,

$$\begin{aligned}((e, i_0, j_0)x(f, i_0, j_0), i, j)\theta &= (e, i, j)((e, i_0, j_0)x(f, i_0, j_0))(f, i, j)\\ &= (e, i, j_0)x(f, i_0, j)\\ &= (e, i, j)x(f, i, j).\end{aligned}$$

Finally, we show that θ is a homomorphism. Let $(x, i, j) \in (e, i_0, j_0) \cdot S(f, i_0, j_0) \times B$ and $(y, k, s) \in (g, i_0, j_0)S(h, i_0, j_0) \times B$. Hence,

$$\begin{aligned}(x, i, j)\theta(y, k, s)\theta &= (e, i, j)x(f, i, j)(g, k, s)y(h, k, s)\\ &= (e, i, s)x(f, i, j)(g, k, s)y(h, i, s)\\ &= (e, i, s)(x(f, i_0, j_0))(f, i, j)(g, k, s)((g, i_0, j_0)y)(h, i, s)\\ &= (e, i, s)x(fg, i_0, j_0)y(h, i, s)\\ &= (e, i, s)x(f, i_0, j_0)(g, i_0, j_0)y(h, i, s)\\ &= (e, i, s)xy(h, i, s).\end{aligned}$$

Since $xy \in (e, i_0, j_0)S(f, i_0, j_0)(g, i_0, j_0)S(h, i_0, j_0) \subseteq (e, i_0, j_0)S(h, i_0 j_0)$, $((x, i, j)(y, k, s))\theta = (xy, i, s)\theta = (e, i, s)xy(h, i, s)$. Thus, $((x, i, j)(y, k, s))\theta = (x, i, j)\theta(y, k, s)\theta$.

Let $T = \bigcup \{(e, i_0, j_0)S(f, i_0, j_0) : e, f \in K\}$. Hence, we have shown that $S \cong T \times B$. Clearly, T is a semigroup with $E_T = \{(e, i_0, j_0) : e \in K\} \cong K$. If $a \in (e, i_0, j_0)S(f, i_0, j_0)$ there exists $x \in S$ such that $a = axa = a((f, i_0, j_0)\, x(e, i_0, j_0))a$ and, hence, T is regular. By [1, Theorem 1.17], T is an inverse semigroup. The converse follows by a routine calculation.

COROLLARY. *S is a regular semigroup whose idempotents form a rectangular band B if and only if $S \cong G \times B$ where G is a group. In particular, S is a regular semigroup whose idempotents form a right zero semigroup if and only if S is a right group.*

In the next two remarks, we use the notation of the theorem.

REMARK 1. The theorem is of particular interest in the case S is bisimple. In this case, T has been described completely mod groups for several classes of K (for example, see [3] and [4]). Hence, in these cases, the structure of S may be described completely mod groups.

REMARK 2. We now give another case of particular interest. A semigroup which is a union of groups is called a Clifford semigroup [2, p. 43]. Clearly, a Clifford semigroup is regular. S is a Clifford semigroup if and only if T is a Clifford semigroup [1, p. 130, Exercise 10]. Hence, since in this case the structure of T is known completely mod groups [1, p. 128, Theorem 4.11], the structure of S is also known.

REMARK 3. The theorem represents a natural initial step in the study of the structure of regular semigroups S such that E_S is a subsemigroup beyond the inverse semigroups. We also take a step in the solution of the following problem: If S is a regular semigroup such that E_S is a subsemigroup, how is the structure of S affected by that of E_S?

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