FUNCTIONS OF BOUNDED VARIATION AND MOMENT-SEQUENCES OF CONTINUOUS FUNCTIONS

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This paper gives a necessary and sufficient condition for certain functions to be of bounded variation.

Let \(U_0\) denote (see [1, p. 24]) the class of functions from \([0, 1]\) to the real numbers to which \(f\) belongs only in case \(f(0+)\) exists, \(f(1-)\) exists, and, for each number \(x\) between 0 and 1, \(f(x-)\) exists, \(f(x+)\) exists, and either \(f(x-) \leq f(x) \leq f(x+)\) or \(f(x+) \leq f(x) \leq f(x-)\).

Theorem. If \(f\) is in \(U_0\), then a necessary and sufficient condition that \(f\) be of bounded variation is that, for each moment-sequence \(c\) of a continuous function from \([0, 1]\) to the real numbers, the limit

\[
M_f(c) = \lim_{n \to \infty} \sum_{k=0}^{n} f(k/n) \binom{n}{k} \Delta^{n-k}c_k
\]

exists. Furthermore, if \(f\) is of bounded variation and \(c\) is the moment-sequence of the continuous function \(g\) from \([0, 1]\) to the real numbers, then \(M_f(c) = \int_0^1 fdg\).

This theorem bears some similarity to a theorem of MacNerney [2, p. 368] which states, in terms of the kind of limit described above, a necessary and sufficient condition for a sequence to be the moment-sequence of a function of bounded variation. Also, Tonne [3] has used this kind of limit to “integrate” certain functions with respect to certain sequences in the sense that one might describe \(M_f(c)\) as the integral of \(f\) with respect to \(c\).

Definitions. If each of \(n\) and \(k\) is a nonnegative integer and \(c\) is a number-sequence and \(f\) is a function from \([0, 1]\) to the real numbers, then

\[
\Delta^n c_k = \sum_{q=0}^{n} (-1)^q \binom{n}{q} c_{k+q},
\]

\[
L(f, c)_n = \sum_{p=0}^{n} f(p/n) \binom{n}{p} \Delta^{n-p}c_p,
\]

\(I\) is the identity function on the interval \([0, 1]\), \(B(f)_n\) is the Bernstein polynomial for \(f\) of order \(n\), namely,

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\[ \sum_{p=0}^{n} f(p/n) \binom{n}{p} (1 - I)^{n-p} I^{p}, \]

\( f_0 \left| df \right| \) denotes the total variation of \( f \) on \([0, 1]\) if \( f \) is of bounded variation, and \( c \) is the moment-sequence of \( f \): \( c_p = \int_0^1 f^p df \) (\( p = 0, 1, \cdots \)).

**Proof of Theorem.** Suppose that \( f \) is of bounded variation and \( c \) is the moment sequence of the continuous function \( g \) from \([0, 1]\) to the real numbers. If \( n \) is a positive integer,

\[
L(f, c)_n = \sum_{p=0}^{n} f(p/n) \binom{n}{p} \Delta^{n-p} c_p
\]

\[
= \sum_{p=0}^{n} f(p/n) \binom{n}{p} \int_0^1 \left(1 - I\right)^{n-p} I^{p} dg
\]

\[
= \int_0^1 B(f)_n dg
\]

\[
= - \int_0^1 g dB(f)_n + B(f)_n(1) \cdot g(1) - B(f)_n(0) \cdot g(0)
\]

\[
= - \int_0^1 g dB(f)_n + f(1)g(1) - f(0)g(0).
\]

Since \( f \) is in \( U_0 \), the Bernstein polynomial sequence \( B(f) \) has limit \( f \) on \([0, 1]\) except, perhaps, at countably many points of \([0, 1] \) (see [1, p. 27]). The sequence \( B(f) \) is "uniformly of bounded variation" (see [1, p. 25]). So (see [4, p. 31]) the limit of the sequence \( L(f, c) \) is

\[ - \int_0^1 g dB(f) + f(1)g(1) - f(0)g(0), \]

which is \( f_0 \left| df \right| \). On the other hand, suppose that, for each moment sequence \( c \) of a continuous function from \([0, 1]\) to the real numbers, the limit \( M(f, c) \) exists. Suppose that \( g \) is a continuous function from \([0, 1]\) to the numbers and let \( c \) be its moment-sequence and, for each positive integer \( n \), let \( T_n(g) \) be \( L(f, c)_n - f(1)g(1) + f(0)g(0) \). With the aid of the preceding computation for \( L(f, c)_n \) we see that \( T_n \) is a continuous linear transformation from the space of continuous functions on \([0, 1] \) (with maximum modulus norm) to the real numbers; the norm of \( T_n \) is \( \int_0^1 \left| dB(f)_n \right| \). For each continuous function \( g \) from \([0, 1]\) to the numbers, the sequence \( T(g) \) converges, so that, by the "principle of uniform boundedness," there is a number \( K \) such that if \( n \) is a positive integer then \( \int_0^1 \left| dB(f)_n \right| < K \), so that (see [1, p. 25]) \( f \) is of bounded variation on \([0, 1]\).
References


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