FUNCTIONS OF BOUNDED VARIATION AND MOMENT-SEQUENCES OF CONTINUOUS FUNCTIONS

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This paper gives a necessary and sufficient condition for certain functions to be of bounded variation.

Let U_0 denote (see [1, p. 24]) the class of functions from $[0, 1]$ to the real numbers to which f belongs only in case $f(0+)$ exists, $f(1-)$ exists, and, for each number x between 0 and 1, $f(x-)$ exists, $f(x+)$ exists, and either $f(x-) \leqq f(x) \leqq f(x+)$ or $f(x+) \leqq f(x) \leqq f(x-)$.

THEOREM. *If f is in U_0, then a necessary and sufficient condition that f be of bounded variation is that, for each moment-sequence c of a continuous function from $[0, 1]$ to the real numbers, the limit*

$$M_f(c) = \operatorname*{Lim}_{n\to\infty} \sum_{k=0}^{n} f(k/n) \binom{n}{k} \Delta^{n-k} c_k$$

exists. Furthermore, if f is of bounded variation and c is the moment-sequence of the continuous function g from $[0, 1]$ to the real numbers, then $M_f(c) = \int_0^1 f dg$.

This theorem bears some similarity to a theorem of MacNerney [2, p. 368] which states, in terms of the kind of limit described above, a necessary and sufficient condition for a sequence to be the moment-sequence of a function of bounded variation. Also, Tonne [3] has used this kind of limit to "integrate" certain functions with respect to certain sequences in the sense that one might describe $M_f(c)$ as the integral of f with respect to c.

DEFINITIONS. If each of n and k is a nonnegative integer and c is a number-sequence and f is a function from $[0, 1]$ to the real numbers, then

$$\Delta^n c_k = \sum_{q=0}^{n} (-1)^q \binom{n}{q} c_{k+q},$$

$$L(f, c)_n = \sum_{p=0}^{n} f(p/n) \binom{n}{p} \Delta^{n-p} c_p,$$

I is the identity function on the interval $[0, 1]$, $B(f)_n$ is the *Bernstein polynomial for f of order n*, namely,

Received by the editors May 9, 1967.

$$\sum_{p=0}^{n} f(p/n)\binom{n}{p}(1-I)^{n-p}I^p,$$

$\int_0^1 |df|$ denotes the total variation of f on $[0, 1]$ if f is of bounded variation, and c is the *moment-sequence* of f: $c_p = \int_0^1 I^p df$ $(p=0, 1, \cdots)$.

Proof of Theorem. Suppose that f is of bounded variation and c is the moment sequence of the continuous function g from $[0, 1]$ to the real numbers. If n is a positive integer,

$$\begin{aligned} L(f, c)_n &= \sum_{p=0}^{n} f(p/n)\binom{n}{p}\Delta^{n-p}c_p \\ &= \sum_{p=0}^{n} f(p/n)\binom{n}{p}\int_0^1 (1-I)^{n-p}I^p dg \\ &= \int_0^1 B(f)_n dg \\ &= -\int_0^1 g\, dB(f)_n + B(f)_n(1)\cdot g(1) - B(f)_n(0)\cdot g(0) \\ &= -\int_0^1 g\, dB(f)_n + f(1)g(1) - f(0)g(0). \end{aligned}$$

Since f is in U_0, the Bernstein polynomial sequence $B(f)$ has limit f on $[0, 1]$ except, perhaps, at countably many points of $[0, 1]$ (see [**1**, p. 27]). The sequence $B(f)$ is "uniformly of bounded variation" (see [**1**, p. 25]). So (see [**4**, p. 31]) the limit of the sequence $L(f, c)$ is $-\int_0^1 g\, df + f(1)g(1) - f(0)g(0)$, which is $\int_0^1 f\, dg$.

On the other hand, suppose that, for each moment sequence c of a continuous function from $[0, 1]$ to the real numbers, the limit $M_f(c)$ exists. Suppose that g is a continuous function from $[0, 1]$ to the numbers and let c be its moment-sequence and, for each positive integer n, let $T_n(g)$ be $L(f, c)_n - f(1)g(1) + f(0)g(0)$. With the aid of the preceding computation for $L(f, c)_n$ we see that T_n is a continuous linear transformation from the space of continuous functions on $[0, 1]$ (with maximum modulus norm) to the real numbers; the norm of T_n is $\int_0^1 |dB(f)_n|$. For each continuous function g from $[0, 1]$ to the numbers, the sequence $T(g)$ converges, so that, by the "principle of uniform boundedness," there is a number K such that if n is a positive integer then $\int_0^1 |dB(f)_n| < K$, so that (see [**1**, p. 25]) f is of bounded variation on $[0, 1]$.

References

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