ON THE DERIVATIVE OF AN ENTIRE FUNCTION[1]

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1. **Introduction.** Our present paper has several objectives. First, in §2, we shall establish a new representation for the derivative of an entire function of finite order. This representation is a generalization of one developed in a previous paper [1] but we shall derive it by a method simpler and more direct than that used for the earlier representation.

Secondly, in §3, we shall apply the new representation to the following theorem stated by Laguerre but proved by Borel.

Let f be a real entire function of finite genus p with m nonreal zeros. In addition to one zero of the derivative f' of f between each pair of adjacent real zeros of f, f' has at most $p+m$ real and nonreal zeros.

Our proof of this theorem will be simpler, more direct and much shorter than Borel's eight page proof [2].

Thirdly, in §4, we shall deduce from our representation some theorems on the location of the zeros of f' when the zeros of f lie in two specified domains. These theorems include some generalizations of Lucas' Theorem (on the critical points of polynomials) to entire functions of finite order.

Finally, in §5, we shall apply the theorem of the previous section to a number of special cases in which the domains are rays, sectors, half strips or circular disks.

2. **Representation for the derivative.** In the sequel we denote by $[\rho]$ the largest integer not exceeding ρ. Our basic theorem is then the following.

THEOREM (2.1). *Let f be an entire function of finite order ρ having zeros $b_1, b_2, \cdots, b_m, a_1, a_2, a_3, \cdots$ where $0<|a_1|\leqq|a_2|\leqq\cdots$ and thus*

$$\sum_{j=1}^{\infty} |a_j|^{-p-1} < \infty, \qquad p \leqq [\rho]. \tag{2.1}$$

Let $\zeta_1, \zeta_2, \cdots, \zeta_n$ with $n=m+[\rho]$ be zeros of the derivative f' of f. Then for all $z\neq a_j$

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$$f'(z) = \frac{f(z)\psi(z)}{\phi(z)} \sum_{j=1}^{\infty} \frac{\phi(a_j)}{\psi(a_j)(z-a_j)}, \tag{2.2}$$

where

$$\phi(z) = \prod_{k=1}^{m} (z - b_k), \qquad \psi(z) = \prod_{k=1}^{n} (z - \zeta_k). \tag{2.3}$$

If $\{b_1, b_2, \cdots, b_m\} = \emptyset$, *take* $m=0$ *and* $\phi(z) \equiv 1$ *in* (2.2). *The series in* (2.2) *converges uniformly in every bounded domain.*

PROOF. First, we shall show that

$$F(z) = \frac{f(z)}{\phi(z)} \sum_{j=1}^{\infty} \frac{\phi(a_j)}{\psi(a_j)(z-a_j)} \tag{2.4}$$

is an entire function. We begin by noting that

$$F_N(z) = \frac{f(z)}{\phi(z)} \sum_{j=1}^{N} \frac{\phi(a_j)}{\psi(a_j)(z-a_j)}$$

is an entire function since f/ϕ has simple zeros at the a_j. It remains to show that the sequence $\{F_N\}$ converges uniformly in every bounded domain.

For this purpose, let us choose $R>0$ and $M = \max|f(z)/\phi(z)|$ for $|z| = R$. Let us then choose N so large that

$$|a_j| > 2\max(|R|, |b_1|, \cdots, |b_m|, |\zeta_1|, \cdots, |\zeta_n|) \text{ for } j > N.$$

Hence, for $j>N$ and $|z| \leqq R$

$$|z - a_j| > |a_j|/2, \quad |a_j - b_k| < (3/2)|a_j|, \quad |a_j - \zeta_k| > |a_j|/2,$$
$$|\phi(a_j)| < (3/2)^m |a_j|^m, \qquad |\psi(a_j)| > |a_j|^n 2^{-n}.$$

Thus for all $|z| \leqq R$

$$\begin{aligned} |F(z) - F_N(z)| &\leqq M \sum_{j=N+1}^{\infty} \frac{|\phi(a_j)|}{|\psi(a_j)|\,|z-a_j|} \\ &< M 3^m 2^{n-m} \sum_{j=N+1}^{\infty} |a_j|^{m-n-1}. \end{aligned} \tag{2.5}$$

Since $n = m + [\rho] \geqq m + p$, we infer from (2.1) that, by choosing N sufficiently large, we may make the right side of (2.5) arbitrarily small. That is, $F_N(z) \to F(z)$ uniformly in every bounded domain.

Secondly, we notice that, since at the zeros a_j of f/ϕ

$$F(a_j) = \lim_{z\to a_j} F(z) = f'(a_j)/\psi(a_j), \qquad j = 1, 2, 3, \cdots,$$

we may write

$$[f'(z)/\psi(z)] - F(z) = [f(z)/\phi(z)]g(z), \tag{2.6}$$

where $g(z)$ is an entire function. Let us show that $g(z)\equiv 0$.

For this purpose, we use the Hadamard Theorem to write f in the form

$$f(z) = \phi(z)e^{Q(z)} \prod_{j=1}^{\infty} (1 - z/a_j) \exp[(z/a_j) + \cdots + (1/p)(z/a_j)^p],$$

where $Q(z)$ is a polynomial of degree $q \leqq [\rho]$. Thus

$$\frac{f'(z)}{f(z)} = \frac{\phi'(z)}{\phi(z)} + Q'(z) + \sum_{j=1}^{\infty} \frac{z^p}{a_j^p(z - a_j)}\cdot$$

From this we infer at once that, as $z\to\infty$ with $z\neq a_j$,

$$\lim \left| f'(z)/z^{[\rho]}f(z) \right| = 0. \tag{2.7}$$

Since $|\phi(z)/\psi(z)| = O(|z|^{m-n})$ as $z\to\infty$ and $n=m+[\rho]$, we conclude from (2.7) that

$$\lim \left| f'(z)\phi(z)/f(z)\psi(z) \right| = 0.$$

Also from (2.4)

$$\lim \left| F(z)\phi(z)/f(z) \right| = 0$$

and thus from (2.6), lim inf $g(z)=0$. Hence, g is a bounded entire function and so $g(z)=\text{const}=0$, establishing the representation (2.2).

3. **Laguerre-Borel Theorem.** We shall now prove a version of this theorem stated in terms of the order of the function f, instead of its genus.

THEOREM (3.1). *Let f be a real entire function of finite order ρ having m nonreal zeros b_k. Let set R consist of one real zero of the derivative f' chosen between each pair of adjacent zeros of f. Then f' has at most $n=[\rho]+m$ zeros, real and nonreal, not belonging to R.*

PROOF. Let us assume on the contrary that f' has the set

$$V = (\zeta_1, \zeta_2, \cdots, \zeta_{n+1})$$

of zeros not in R. Then, since $f'(\zeta_{n+1})=0$, we obtain from (2.2) and (2.3) the identity

$$\sum_{j=1}^{\infty} \frac{\phi(a_j)}{(\zeta_1 - a_j)(\zeta_2 - a_j) \cdots (\zeta_{n+1} - a_j)} = 0. \tag{3.1}$$

Also, the m nonreal zeros of f must consist of $(m/2)$ conjugate imaginary pairs $(b_k, \bar{b}_k)$ since f is real. Hence,

$$\phi(a_j) = \prod_{k=1}^{m/2} (b_k - a_j)(\bar{b}_k - a_j) > 0, \qquad \text{for all } a_j. \tag{3.2}$$

Let us separate V into three subsets, any one of which might be empty:

(1) V_1 consisting of all real ζ_k lying on $\mathcal{H}(R)$, the smallest interval of the real axis containing R;

(2) V_2 consisting of all real ζ_k not lying on $\mathcal{H}(R)$;

(3) V_3 consisting of all nonreal ζ_k.

Let us suppose that $\zeta_1 \in V_1 \cap I_j$ where I_j is the interval between the consecutive real zeros a_j and a_{j+1} of f, $a_j < a_{j+1}$. By Rolle's Theorem f' has on I_j an odd number of zeros of which just one belongs to R. Hence, either f' has also a zero $\zeta_2 \in V_1 \cap I_j$, $\zeta_2 \neq \zeta_1$, or such a ζ_2 can be introduced into V in place of some ζ_k originally in V. Thus, without loss of generality, we can assume that

$$V_1 \cap I_j = (\zeta_1, \zeta_2, \cdots, \zeta_{2k}), \qquad k \geqq 1.$$

The corresponding factor in the denominators of (3.1) is

$$(\zeta_1 - a_j)(\zeta_2 - a_j) \cdots (\zeta_{2k} - a_j) > 0, \qquad j = 1, 2, \cdots.$$

Let us next suppose $\zeta_\mu \in V_2$. The corresponding factor in the denominators of (3.1) is $(\zeta_\mu - a_j)$ which has the same sign for all j.

Finally, let us suppose $\zeta_\nu \in V_3$. Since f' is real, it has also $\bar{\zeta}_\nu$ as a zero. If $\bar{\zeta}_\nu \notin V_3$, we may admit $\bar{\zeta}_\nu$ to V in place of some other ζ_k originally in V. The corresponding factor in the denominators of (3.1) is

$$(\zeta_\nu - a_j)(\bar{\zeta}_\nu - a_j) > 0, \qquad j = 1, 2, \cdots.$$

In short, if f' had $n+1$ zeros not in R, every term in (3.1) would have the same sign and so the sum could not vanish, in contradiction to (3.1). Hence, Theorem (3.1) is true.

4. **Entire functions with zeros in two domains.** In the sequel we shall use the notation of $\mathcal{C}(T)$ for the complement of a set T in the complex plane, $\mathcal{H}(T)$ for the convex hull of T and $\mathcal{S}(T, \nu)$ for the set of points from which T subtends an angle of at least ν. The set $\mathcal{S}(T, \nu)$ is star shaped relative to T and has the properties

$$T \subset \mathcal{H}(T) = \mathcal{S}(T, \pi) \subset \mathcal{S}(T, \nu_1) \subset \mathcal{S}(T, \nu_2)$$

if $\pi > \nu_1 > \nu_2 \geqq 0$.

As an application of Theorem (2.1), we shall prove the following:

THEOREM (4.1). *Let f be an entire function of finite order ρ which has the zeros $a_1, a_2, a_3, \cdots$ on a set T for which $\mathcal{C}[\mathcal{H}(T)]$ is not empty. Let f also have the zeros $b_1, b_2, \cdots, b_m$ in $\mathcal{C}[\mathcal{S}(T, \beta)]$ with $0 < \beta < \pi/m$. Then the derivative f' of f has at most $n = m + [\rho]$ zeros in $\mathcal{C}[\mathcal{S}(T, \nu)]$ where $\nu = (\pi - m\beta)/(n+1)$.*

PROOF. Let us suppose that f' had the $n+1$ distinct zeros $\zeta_1, \zeta_2, \cdots, \zeta_{n+1}$ in $\mathcal{C}[\mathcal{S}(T, \nu)]$. Since T subtends an angle less than ν at each ζ_k, we can associate with each ζ_k a point ξ_k, $\xi_k \neq \zeta_k$, so that

$$(4.1) \qquad 0 \leqq \arg[(\xi_k - \zeta_k)/(a_j - \zeta_k)] < \nu, \qquad \text{for all } j \text{ and } k.$$

Likewise, we can associate with each b_k a point $c_k \neq b_k$ such that

$$(4.2) \qquad 0 \leqq \arg[(a_j - b_k)/(c_k - b_k)] < \beta \qquad \text{for all } j \text{ and } k.$$

If we multiply (3.1) by

$$[(\xi_1 - \zeta_1) \cdots (\xi_{n+1} - \zeta_{n+1})]/[(c_1 - b_1) \cdots (c_m - b_m)],$$

we may rewrite (3.1) as

$$(4.3) \quad \sum_{j=1}^{\infty} \frac{(a_j - b_1) \cdots (a_j - b_m)(\xi_1 - \zeta_1) \cdots (\xi_{n+1} - \zeta_{n+1})}{(c_1 - b_1) \cdots (c_m - b_m)(a_j - \zeta_1) \cdots (a_j - \zeta_{n+1})} = 0.$$

According to (4.1) and (4.2), each term on the left side of (4.3) is a vector lying in the sector

$$0 \leqq \arg z \leqq m\beta + (n+1)\nu < \pi.$$

Therefore, the sum of these vectors cannot vanish, in contradiction to (4.3). Hence, f' can have at most n zeros in $\mathcal{C}[\mathcal{S}(T, \nu)]$ as stated in Theorem (4.1).

In Theorem (4.1) we tacitly assumed that $m \geqq 1$. If however $\{b_1, b_2, \cdots, b_m\} = \varnothing$, we may similarly deduce from Theorem (2.1) the following result (Marden [**5**]).

THEOREM (4.2). *Let all the zeros of an entire function f of finite order lie on a set T with $\mathcal{C}[\mathcal{H}(T)] \neq \varnothing$. The derivative f' then has at most $n = [\rho]$ zeros in $\mathcal{C}[(\mathcal{S}T, \pi/(n+1))]$.*

In particular, if $0 \leqq \rho < 1$, then $n = 0$ and all zeros of f' lie in $\mathcal{H}(T)$. Hence, Theorem (4.2) is a generalization of Lucas' Theorem to entire functions.

5. **Special cases.** Let us specialize in several ways the set T occurring in Theorem (4.1).

Let us suppose that T is the positive real axis. Then $S(T, \nu)$ is the sector $|\arg z| \leqq \pi - \nu$ and hence $\mathcal{C}[\mathcal{S}(T, \nu)]$ is the sector $|\arg(-z)| < \nu$. Thus, we obtain

COROLLARY (5.1). *Let f, an entire function of finite order ρ, have m zeros in the sector $|\arg(-z)| < \beta < \pi/m$ and its remaining zeros on the positive real axis. Then f' has at most $n = m + [\rho]$ zeros in the sector $|\arg(-z)| < \nu$, where $\nu = (\pi - m\beta)/(n+1)$.*

An interesting special case is the one in which f has m negative zeros and all its remaining zeros positive. In that case $\beta = 0$, $\nu = \pi/(n+1)$. We note that in this special case f is not necessarily a real function.

As a generalization of Corollary (5.1) let us choose T as the sector

$$|\arg z| \leqq \alpha < \nu/2.$$

Then $\mathcal{S}(T, \nu)$ is the sector $|\arg z| \leqq \pi - \nu + \alpha$ so that $\mathcal{C}[\mathcal{S}(T, \nu)]$ is the sector $|\arg(-z)| < \nu - \alpha$. Thus, we obtain

COROLLARY (5.2). *Let f, an entire function of finite order ρ, have m zeros in the sector $|\arg(-z)| < \beta < \pi/m$ and its remaining zeros in the sector $|\arg z| \leqq \alpha < \nu/2$, where $\nu = (\pi - m\beta)/(n+1)$ and $n = m + [\rho]$. Then f' has at most n zeros in the sector $|\arg(-z)| < \nu - \alpha$.*

An additional special case of interest is the one for which T is a half-strip such as

$$(4.4) \qquad |\mathcal{I}(z)| \leqq c, \qquad \mathcal{R}(z) \geqq a > 0.$$

Let C_β be the circle containing the arc from which the segment between points $z_1 = a + ic$, $z_2 = a - ic$ subtends an angle of β. Let $z_3 = b + ic$ and $z_4 = b - ic$ be also points on C_β. Let K_β consist of the points exterior to C_β but interior to the angle bounded by lines z_1, z_4 and z_2, z_3 but not containing T. We may thus state the following:

COROLLARY (5.3). *If f, an entire function of finite order ρ, has m zeros in the domain K_β and its remaining zeros in the half-strip* (4.4), *then f' has at most $n = m + [\rho]$ zeros in the domain K_ν, where $\beta < \pi/m$, $\nu = (\pi - \beta m)/(n+1)$.*

Finally, let us consider the special case of an entire function with a finite number of zeros. For this case, we may state the following:

COROLLARY (5.4). *If f, an entire function of finite order ρ, has m zeros* *in the domain* $|z|>a \csc(\beta/2)$, *where* $\beta<\pi/m$, *and its remaining zeros in the disk* $|z|\leqq a$, *then* f' *has at most* $n=m+[\rho]$ *zeros in the domain* $|z|>a \csc(\nu/2)$, *where* $\nu=(\pi-m\beta)/(n+1)$.

Corollary (5.4) has some similarity to the following result due to Walsh [3]. If the polynomial f of degree m_1+m_2 has m_1 zeros in region C_1: $|z|\leqq r_1$ and m_2 zeros in region C_2: $|z|\geqq r_2$, then its derivative f' has all its zeros in C_1, C_2 and C: $|z|\geqq r$, $=(m_1r_2-m_2r_1)/(m_1+m_2)$; furthermore, if $r_1<\text{Min}(r, r_2)$, f' has exactly m_1-1 zeros in C_1 and m_2 zeros in $C\cup C_2$.

It has also some similarity to the following result of Marden [4]: If an nth degree polynomial has k zeros $(1<k\leqq n)$ in the region C: $|z|\leqq R$, its derivative has at most $n-k$ zeros in region $|z|>R \csc[\pi/2(n-k+1)]$.

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