WEAKLY ALMOST PERIODIC FUNCTIONS AND FOURIER-STIELTJES TRANSFORMS

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Let G be an infinite nondiscrete abelian group; Γ the (noncompact) dual group of G; $M(G)$ the algebra of bounded Borel measures on G; $M(G)^\wedge$ the algebra of Fourier-Stieltjes transforms; $M(G)^{\wedge -}$ the completion of $M(G)^\wedge$ in the sup-norm topology on Γ; and $\mathrm{WAP}(\Gamma)$ the algebra of continuous bounded weakly almost periodic functions on Γ.

The object of this paper is to show the following theorem.

THEOREM. *Let* Γ *be an infinite noncompact abelian group. Then* $M(G)^{\wedge -} \neq \mathrm{WAP}(\Gamma)$.

PROOF. We consider first the case that Γ is discrete. If Γ is not of bounded order, then Rudin [4] using a deep trigonometric inequality has shown this result. A proof based on an elementary inequality may be found in [3].

Thus we may assume that Γ is of bounded order. Let $Z(p)$ denote the finite cyclic group of p elements of unimodular complex numbers; and $Z(p)^\infty$ the weak direct product of $Z(p)$ over a countable infinite index set. Thus there exists p such that $Z(p)^\infty$ is a subgroup of Γ, [1, p. 449]. We may assume that $\Gamma = Z(p)^\infty$.

There exists $\lambda_n \in M(\Gamma)$ such that $\|\lambda_n\| = 1$ and $\|\lambda_n^\wedge\|_\infty \leqq 1/n$, $n = 1, 2, \cdots$. Let S_n denote the supp λ_n. We may assume that the S_n's are finite sets and pairwise disjoint e.g. [2, Theorem 3.2]. Let f_0 be a (continuous) bounded function on Γ such that $\int_\Gamma f_0 d\lambda_n = \|\lambda_n\| = 1$, $\|f_0\|_\infty \leqq 1$, and $\operatorname{supp} f_0 = \bigcup_{n=1}^\infty S_n$.

Let g be a (continuous) bounded function on Γ. $g \in M(G)^{\wedge -}$ if and only if $\{\lambda_n\} \subset M(\Gamma)$, $\|\lambda_n\| \leqq 1$, and $\lambda_n^\wedge(x) \xrightarrow{n} 0$ for all $x \in G$ implies $\int_\Gamma g d\lambda_n \xrightarrow{n} 0$, [2, Theorem 1.9]. Thus $f_0 \notin M(G)^{\wedge -}$. It remains to show that we may pick f_0 such that $f_0 \in \mathrm{WAP}(\Gamma)$.

Let $S = \bigcup_{n=1}^\infty S_n$. It is enough to construct S such that $S \cap (S+y)$ is finite for every $y \neq 0$ since Rudin [4] has shown that any continuous

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bounded function f with $\operatorname{supp} f = S$ would then be weakly almost periodic. A proof using quasi-uniform convergence may be found in [3].

Let $H \subset \Gamma$ be a finite set. Let $\alpha(H)$ [$\beta(H)$] denote the last [first] coordinate such that all elements of H are 1 for coordinates $< \alpha(H)$ [$> \beta(H)$]. Let the S_n's be constructed such that $\alpha(S_1) < \beta(S_1) < \alpha(S_2) < \beta(S_2) \cdots < \alpha(S_n) < \beta(S_n)$. Thus $f_0 \in \mathrm{WAP}(\Gamma)$.

Now let Γ be any infinite noncompact abelian group. If Γ contains a copy of R^n then [4] applies. If not, then the structure theorem for locally compact abelian groups [1, p. 389] implies that Γ contains a compact open subgroup Λ. Γ/Λ is infinite and discrete. Let f be the function on Γ/Λ given by the preceding case. Finally, extend f canonically to Γ. That $f \notin M(G)^{\wedge -}$ follows from the characterization of $M(G)^{\wedge -}$, [2, Theorem 1.9].

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