A CHARACTERIZATION OF PURE STATES OF C^*-ALGEBRAS

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If ρ is a state of a C^*-algebra $\mathfrak{A}$ then its *definite set* $\mathfrak{D}_\rho$ is the set of selfadjoint operators A in $\mathfrak{A}$ such that $\rho(A^2)=\rho(A)^2$. Kadison and Singer [**3**, Theorem 4] showed that a state of all bounded operators $\mathfrak{B}(\mathfrak{H})$ on a Hilbert space $\mathfrak{H}$ is pure if and only if its definite set is maximal (in the set of definite sets due to states ordered by inclusion), and they left the problem open for states on general C^*-algebras. Of course, if there exists a representation of $\mathfrak{A}$ onto the complex numbers then $\mathfrak{A}_{SA}$—the selfadjoint operators in $\mathfrak{A}$—is itself the definite set of some state, so unless $\mathfrak{A}$ is in this case abelian, there will exist pure states whose definite sets are not maximal. We shall in the present note solve the problem to the affirmative if $\mathfrak{A}$ has no one-dimensional representations.

THEOREM. *Let $\mathfrak{A}$ be a C^*-algebra with identity and with no one-dimensional representations. Then a state of $\mathfrak{A}$ is pure if and only if its definite set is maximal.*

Kadison and Singer showed [**3**, proof of Theorem 4] that if $\mathfrak{J}_\rho$ denotes the left kernel of the state ρ, viz. $\mathfrak{J}_\rho=\{A\in\mathfrak{A}: \rho(A^*A)=0\}$, then $\mathfrak{D}_\rho=\boldsymbol{R}I+\mathfrak{K}_{SA}$, where $\mathfrak{K}=\mathfrak{J}_\rho\cap\mathfrak{J}_\rho^*$, and where I is the identity of $\mathfrak{A}$, $\boldsymbol{R}$ the real numbers. Moreover they showed that if $\mathfrak{D}_\rho$ is maximal then ρ is pure. Hence, in order to prove the theorem it suffices to prove the following

LEMMA. *Let $\mathfrak{A}$ be a C^*-algebra. Let ρ be a pure state of $\mathfrak{A}$. If ω is a state of $\mathfrak{A}$ for which $\mathfrak{D}_\omega\supset\mathfrak{D}_\rho$ then either ω is a homomorphism, so $\mathfrak{D}_\omega=\mathfrak{A}_{SA}$, or $\mathfrak{D}_\omega=\mathfrak{D}_\rho$.*

PROOF. We may assume $\mathfrak{A}$ has an identity denoted by I. Let $\rho=\omega_x\circ\pi_\rho$, where π_ρ is an irreducible representation of $\mathfrak{A}$ on a Hilbert space, and x is a unit vector [**2**, 2.5.4]. Let $\mathfrak{J}=$kernel π_ρ, and suppose $\omega(\mathfrak{J})\neq 0$. Let $\omega=\omega_y\circ\pi_\omega$. Then $\mathfrak{F}=\pi_\omega(\mathfrak{J})$ is a nonzero two-sided ideal in $\pi_\omega(\mathfrak{A})$ such that ω_y restricted to $\mathfrak{F}$ is a homomorphism. Now $\mathfrak{F}^-$ is an ideal in $\pi_\omega(\mathfrak{A})^-$, where the bar denotes weak closure, so there is a central projection E in $\pi_\omega(\mathfrak{A})^-$ such that $\mathfrak{F}^-=E\pi_\omega(\mathfrak{A})^-$ [**1**, p. 45]. By continuity ω_y is a homomorphism of $E\pi_\omega(\mathfrak{A})^-$ and 0 on $(I-E)\pi_\omega(\mathfrak{A})^-$; hence ω_y is a homomorphism on $\pi_\omega(\mathfrak{A})^-$, and ω is a homomorphism on $\mathfrak{A}$. We

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may therefore assume $\mathfrak{J}\subset$ kernel π_ω. Thus ω induces a state $\bar{\omega}$ of $\pi_\rho(\mathfrak{A})$ such that the definite set of $\bar{\omega}$ contains that of ω_x, and $\omega=\bar{\omega}\circ\pi_\rho$. We may therefore assume $\mathfrak{A}$ is irreducible acting on a Hilbert space $\mathfrak{H}$, $\rho=\omega_x$, and $\mathfrak{D}_\omega\supset\mathfrak{D}_{\omega_x}$. Let $\mathfrak{K}=\mathfrak{J}_{\omega_x}\cap\mathfrak{J}_{\omega_x}^*$. Then $\mathfrak{K}$ is a C^*-algebra, and $\mathfrak{K}_{SA}=(\mathfrak{J}_{\omega_x})_{SA}$. Since $\omega|\mathfrak{K}$ is a homomorphism there are two cases. If $\omega(\mathfrak{K})=0$ then if $A\in\mathfrak{J}_{\omega_x}$, $A^*A\in\mathfrak{K}$ so $\omega(A)=0$. Since ω_x is the unique state of $\mathfrak{A}$ which annihilates $\mathfrak{J}_{\omega_x}$ [**3**, 2.9.5], $\omega=\omega_x$ in this case. We may therefore assume $\omega|\mathfrak{K}$ is a nonzero homomorphism. Now $\omega=\omega_y\circ\pi$ with π a representation of $\mathfrak{A}$ on a Hilbert space $\mathfrak{H}_\omega$. If dim $\mathfrak{H}_\omega=1$, ω is a homomorphism of $\mathfrak{A}$. Assume dim $\mathfrak{H}_\omega\geqq 2$. Let $\mathfrak{L}=\pi(\mathfrak{J}_{\omega_x})$. Then $\mathfrak{L}$ is a left ideal in $\pi(\mathfrak{A})$. Let A be a positive operator in $\mathfrak{L}$, i.e. $A\in\mathfrak{L}^+$. Then there is B in $\mathfrak{J}_{\omega_x}$ such that $\pi(B)=A$. Hence $\pi(B^*B)=A^2\in\pi(\mathfrak{K}_{SA})$. But this set is uniformly closed since $\mathfrak{K}$ is a C^*-algebra. Hence $A=(A^2)^{1/2}\in\pi(\mathfrak{J}_{\omega_x}^+)$. Let $\mathfrak{L}_1$ be the uniform closure of $\mathfrak{L}$. Let $A\in\mathfrak{L}_1^+$. Choose S_n in $\mathfrak{L}$ such that $S_n\rightarrow A^{1/2}$ uniformly. Then $S_n^*S_n\in\mathfrak{L}$ and $S_n^*S_n\rightarrow A$ uniformly. By the preceding $S_n^*S_n\in\pi(\mathfrak{J}_{\omega_x}^+)$, a uniformly closed set. Thus $A\in\pi(\mathfrak{J}_{\omega_x}^+)$, and $\mathfrak{L}^+=\mathfrak{L}_1^+$. Thus $\pi(\mathfrak{K})=\mathfrak{L}_1\cap\mathfrak{L}_1^*$, and $\pi(\mathfrak{K})$ has the property that if $A\in\pi(\mathfrak{A})$, $B\in\pi(\mathfrak{K})$ then $BAB\in\pi(\mathfrak{K})$. From the Kaplansky Density Theorem and the strong continuity of multiplication on bounded sets $\pi(\mathfrak{K})^-$ has the same property in $\pi(\mathfrak{A})^-$. As on [**1**, p. 45] there is a projection P in $\pi(\mathfrak{A})^-$ such that $\pi(\mathfrak{K})^-=P\pi(\mathfrak{A})^-P$ (for this see also [**4**, Proposition 3]). But $\omega_y|\pi(\mathfrak{K})$ is a homomorphism; hence so is ω_y on $\pi(\mathfrak{K})^-$. In particular $\omega_y(P)=1$, and ω_y is a pure state on $\pi(\mathfrak{A})$. Since y is cyclic, $\pi(\mathfrak{A})$ is irreducible; hence P is the one-dimensional projection on the subspace generated by y. Let z be a unit vector in $\mathfrak{H}_\omega$ orthogonal to y. Then for $A\in\mathfrak{J}_{\omega_x}$, $\pi(A)z=0$, and $\mathfrak{L}_1\subset\mathfrak{J}_{\omega_z}$. Therefore,

$$\mathfrak{J}_{\omega_x}\subset\pi^{-1}(\mathfrak{L})\subset\pi^{-1}(\mathfrak{J}_{\omega_z}).$$

But $\mathfrak{F}=\pi^{-1}(\mathfrak{J}_{\omega_z})$ is a uniformly closed left ideal in $\mathfrak{A}$. Since $\mathfrak{J}_{\omega_x}$ is a maximal left ideal in $\mathfrak{A}$ there are two possibilities. If $\mathfrak{F}=\mathfrak{A}$ then $\mathfrak{J}_{\omega_z}=\pi(\mathfrak{F})=\pi(\mathfrak{A})$, which is impossible. Therefore $\mathfrak{F}=\mathfrak{J}_{\omega_x}$. Hence $\omega_z\circ\pi(\mathfrak{J}_{\omega_x})=\omega_z(\mathfrak{J}_{\omega_z})=0$, so $\omega_z\circ\pi=\omega_x$. In particular ω_z and ω_x are unitarily equivalent, so we may assume $\mathfrak{H}_\omega=\mathfrak{H}$ and that there is a unitary operator U such that $\omega_{Uz}=\omega_x$. Since this holds for every unit vector z orthogonal to y, dim $\mathfrak{H}=2$. Since furthermore $\omega=\omega_y\circ\pi=\omega_{Uy}$ is a state on $\mathfrak{K}$, Uy is orthogonal to x, so $\mathfrak{D}_\omega=\mathfrak{D}_{\omega_x}$ is the diagonal 2×2 matrices. The proof is complete.

In the course of the proof we showed

COROLLARY. *Let $\mathfrak{A}$ be a C^*-algebra and ρ a pure state of $\mathfrak{A}$ such that* dim $\mathfrak{H}_\rho\geqq 3$, *where $\rho=\omega_x\circ\pi_\rho$, π_ρ being an irreducible representation of*

$\mathfrak{A}$ on a Hilbert space $\mathfrak{H}_\rho$ and x a unit vector in $\mathfrak{H}_\rho$. If ω is a state of $\mathfrak{A}$ such that its definite set contains that of ρ then either ω is a homomorphism or $\omega=\rho$.

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