A NOTE ON FIXED POINT THEOREMS FOR A FAMILY OF NONEXPANSIVE MAPPINGS

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1. **Introduction.** In this note we prove the two following theorems:

THEOREM 1. *Let X be a Banach space and K a nonempty closed convex subset of X. Let $\mathfrak{F}$ be a commuting family of nonexpansive mappings from K into itself and M a compact subset of X such that there exist an $f_1 \in \mathfrak{F}$ and an $x_0 \in K$ satisfying the following properties:*

(i) $\{f_1^n(x_0)\}$ *is a bounded set,*

(ii) $\mathrm{cl}\{f_1^n(x_0)\} \cap M \neq \emptyset$ *for every $x \in K$. Then the family $\mathfrak{F}$ has a common fixed point in M.*

THEOREM 2. *Let X be uniformly convex Banach space and K a nonempty closed convex subset of X. Let $\mathfrak{F}$ be a commuting family of nonexpansive mappings from K into itself and M a bounded subset of X such that there exist an $f_1 \in \mathfrak{F}$ and an $x_0 \in K$ satisfying the following properties:*

(iii) $\{f_1^n(x_0)\}$ *is a bounded set,*

(iv) $\mathrm{cl\ co}\{f_1^n(x)\} \cap M \neq \emptyset$ *for every $x \in K$. Then the family $\mathfrak{F}$ has a common fixed point in M.*

Theorem 1 is a generalization of Theorem 1 in [1] of L. P. Belluce and W. A. Kirk where K is a bounded set. Similarly, Theorem 2 is a generalization of Theorem 2 in [2] of F. E. Browder.

2. **Definition and notations.** Let X be a Banach space. A mapping f from a subset A of X into itself is nonexpansive if $\|f(x)-f(y)\| \leqq \|x-y\|$, for every $x, y \in A$. $f^n(x)$ is defined inductively as $f[f^{n-1}(x)]$, and hence $\{f^n(x_0)\}$ the set of iterate images of x_0. We denote the diameter of a set A by $d(A)$, the closure and the closure convex by $\mathrm{cl}(A)$ and $\mathrm{cl\ co}(A)$ respectively.

The proof of Theorem 1 is in the general line of argument of L. P. Belluce and W. A. Kirk in [1]. Theorem 2 can be seen as a corollary of Theorem 2 in [2].

PROOF OF THEOREM 1. Suppose that the set $\{f_1^n(x_0)\}$ be bounded by the number d. Let B_n denote the closed ball of center $f_1^n(x_0)$ and radius d. We define: $D_k = \bigcap_k^\infty (B_n \cap K)$ and $D = \mathrm{cl}(\bigcup_0^\infty D_k)$.

Then one can show that D is a nonempty closed and bounded convex set which is mapped into itself by the mapping f_1. Applying

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Theorem 1 in [1] to the case where $\mathfrak{F} = \{f_1\}$, we can get a fixed point of f_1 in M. The condition (ii) implies that every fixed point of f_1 must be in M. Hence, the set H_1 of all fixed points of f_1 is a nonempty closed compact subset of M. Furthermore, by the commutativity property of the family $\mathfrak{F}$, $f(H_1) \subset H_1$ for every $f \in \mathfrak{F}$. Also, by compactness of H_1 and by Zorn's lemma, there is a set H^+ which is minimal with respect to being nonempty, compact subset of H_1 and mapped into itself by every $f \in \mathfrak{F}$. Since for every f, $g \in \mathfrak{F}$ we have

$$g[f(H^+)] = f[g(H^+)] \subset f(H^+);$$

therefore $f(H^+)$ is a nonempty compact subset of H_1 and mapped into itself by each $g \in \mathfrak{F}$. Thus, by minimality of H^+, $f(H^+) = H^+$ for every $f \in \mathfrak{F}$. Let C be the set defined as follows:

$$C = \{x \in K \mid \|x - y\| \leqq d(H^+) \text{ for every } y \in H^+\}.$$

Since H^+ is a nonempty set, C is a nonempty closed bounded convex subset of K. Furthermore, $f(H^+) = H^+$ implies that $f(C) \subset C$, for every $f \in \mathfrak{F}$. As a consequence of Theorem 1 in [1], the family $\mathfrak{F}$ has a common fixed point in C. By the condition (ii), this common fixed point must lie in M.

PROOF OF THEOREM 2. By the same argument as in the proof of Theorem 1, the set H_1 of all fixed points of f_1 is a nonempty closed subset of M and hence, also a bounded set. Furthermore, by the uniform convexity of the space X, the set H_1 is a convex set. Also, the commutativity property of the family $\mathfrak{F}$ implies that $f(H_1) \subset H_1$ for every $f \in \mathfrak{F}$. As a consequence of Theorem 2 in [2], the family $\mathfrak{F}$ has a common fixed point in H_1 and hence in M.

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