NOTE ON A THEOREM BY SEN

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In 1966, R. N. Sen published the following theorem [**1**, Theorem 3, p. 882]:

The coordinates of any conformally Euclidean space of class one may be so chosen that its metric assumes the normal form

$$ds^2 = \sum_i (dx^i)^2/[f(\theta)]^2, \qquad \theta = \sum_i (x^i)^2,$$

where f is any real analytic function of θ subject to the restriction

$$(n-1)ff' + \theta ff'' - (n-1)\theta f'^2 \neq 0 \qquad (f' = df/d\theta \text{ etc.}).$$

It is the object of this note to give a disproof of Sen's result. To do so, we use a result of H. Shapiro ([**2**], or see [**3**, pp. 43–51]) who showed that a conformally Euclidean space with metric

$$ds^2 = \sum_i (dy^i)^2/[g(y^1)]^2$$

where g is any real analytic function of y^1, is of class one. The following theorem should now complete the disproof:

THEOREM. *Let*

$$\text{(A)} \qquad ds^2 = \sum_{i=1}^n (dy^i)^2/[g(y^1)]^2$$

and

$$\text{(B)} \qquad ds^2 = \sum_{i=1}^n (dx^i)^2/[f(\theta)]^2, \qquad \theta = \sum_{i=1}^n (x^i)^2,$$

be two quadratic differential forms with $n \geqq 3$, *where* f, g *are arbitrary differentiable functions. Then the necessary and sufficient conditions for a transformation*

$$T: y^i = y^i(x^1, x^2, \cdots, x^n) \qquad (i = 1, \cdots, n)$$

to exist taking (A) *into* (B) *is that*

(1) $f(t) = A_1t + A_2$,

(2) $g(t) = B_1t + B_2$, *and*

(3) $4A_1A_2 + B_1^2 = 0$, *where* A_1, A_2, B_1, B_2, *are constants.*

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PROOF. The sufficiency is obvious, since conditions (1) and (2) ensure that (A) and (B) are the metrics of Riemannian spaces of constant curvature ((A) the Beltrami form and (B) the Riemann form), while condition (3) ensures that they have the same curvature. Thus (A) and (B) are the metrics of equivalent spaces and hence a transformation must exist taking (A) into (B).

To show necessity, let us assume that such a transformation T exists. It thus establishes a conformal equivalence between the (Euclidean) metrics $\sum_{i=1}^{n}(dy^i)^2$ and $\sum_{i=1}^{n}(dx^i)^2/[f(\theta)/g(y^1)]^2$. Hence, by Liouville's theorem, T must be of the form

$$y^i = \frac{a^i\theta + \sum_j b^i_j x^j + c^i}{a\theta + \sum_j b_j x^j + c} \qquad (i = 1, \cdots, n)$$

with

$$\sum_i (dy^i)^2 = \frac{\gamma^2 \sum_i (dx^i)^2}{[a\theta + \sum_j b_j x^j + c]^2}$$

where the a's, b's, c's and γ are constants (i.e. a conformal transformation in which spheres are preserved).

Let $a^1\theta + \sum_j b^1_j x^j + c^1 = M$ and $a\theta + \sum_j b_j x^j + c = N$. Then (A) becomes

$$ds^2 = \frac{\sum_i (dx^i)^2}{(N^2/K^2)[g(M/N)]^2}$$

where K is a constant. Equating this with (B) we obtain the identity

(C) $$f(\theta) \equiv (N/K)g(M/N).$$

Differentiate this expression with respect to x^i

$$2x^i f' \equiv \frac{1}{K}\left[(2ax^i + b_i)g + \frac{N(2a^1x^i + b^1_i) - M(2ax^i + b_i)}{N} g'\right]$$

where $f' \equiv df/d\theta$, $g' \equiv dg/d(M/N)$ or, rearranging terms, and putting $t = M/N$,

$$2x^i(Kf' - ag - a^1g' + ag't) \equiv gb_i + g'b^1_i - g'b_i t \quad (i = 1, \cdots, n).$$

Now, if the right-hand side is zero for some i, then $Kf' - ag - a^1g' + ag't$

$=0$ and so the right-hand side is zero for all i.

If the right-hand side is not zero for any i, then neither is the left-hand side, so that

$$2x^i = \frac{gb_i + g'b_i^1 - g'b_i t}{Kf' - ag - a^1g' + ag't} \qquad (i = 1, \cdots, n),$$

i.e., $x^i = F^i(t, \theta)(i=1, \cdots, n)$.

This would imply that C_n can be identified with a two-dimensional manifold of itself, in contradiction to the fact that the x^i are independent ($n \geqq 3$). Thus we must have

I. $gb_i + g'b_i^1 - g'b_i t = 0 \ (i=1, \cdots, n)$,

II. $Kf' - ag - a^1g' + ag't = 0$.

Integrating I yields (since not all the b_i, b_i^1, can be zero in T)

$$g(t) = B_1 t + B_2, \qquad B_1, B_2 \text{ constants},$$

and substituting this into II gives us $f' = A_1$ so that

$$f(t) = A_1 t + A_2 \qquad (A_1, A_2 \text{ constants}).$$

Thus equations (1) and (2) are established and so (A) and (B) represent spaces of constant curvature. Condition (3) follows when we equate their curvatures.

Thus applying the above theorem to Shapiro's metric, we obtain the result that

$$ds^2 = \sum_{i=1}^{n} (dy^i)^2/[g(y^1)]^2 \qquad (n \geqq 3)$$

where g is any nonlinear, analytic function, is the metric of a conformally Euclidean space of class one which cannot be transformed into Sen's form.

REMARK. Beginning at equation (C), the following alternative proof was suggested by the referee, to whom go my thanks. Its advantages are that it is geometrical in nature and that it eliminates the restrictions of differentiability on the functions f and g:

(C) $$f(\theta) = (N/K)g(M/N).$$

The right side is a homogeneous function of degree 1 in M and N, which does not vanish for geometrical reasons. For any value z in the range of $(N/K)g(M/N)$, the inverse image F_z of z is a union of $(n-2)$-spheres, all located in hyperplanes perpendicular to a fixed line L on which their centers lie. The line contains the centers of the hyperspheres whose equations are $M=0$, $N=0$ resp. (if these two centers coincide, then L collapses to a point). F_z is invariant under rotations about L. The inverse images $f^{-1}(z)$ are unions of hyperspheres; as these equal the F_z, the center 0 of these must lie on L. Hence, there are numbers p, q, r such that any one of the spheres with equations $M=p$, $N=q$, $\theta=r$ passes through the intersection of the other two. That is, there are numbers A, B such that $\theta-r=A(M-p)+B(N-q)$, or, for suitable C, $\theta=AM+BN+C$. Hence, (C) becomes

$$f(AM+BN+C) = (N/K)g(M/N).$$

Thus, in the MN-plane, $(M, N)\to f(AM+BN+C)$ is homogeneous of degree 1, and constant on a parallel family of hyperplanes; hence linear homogeneous:

$$f(AM+BN+C) = A_1(AM+BN),$$

i.e.,

$$f(t) = A_1(t-C) = A_1t + A_2,$$

$$g(M/N) = (K/N)A_1(AM+BN) = AKA_1(M/N) + BKA_1,$$

i.e.,

$$g(t) = AKA_1t + BKA_1 = B_1t + B_2.$$

Thus equations (1) and (2) are established.

References

1. R. N. Sen, *On conformally-flat Riemannian space of class one*, Proc. Amer. Math. Soc. **17** (1966), 880–883.

2. H. Shapiro, *Über die Metrik der subprojektiven Räume*, Trudy Sem. Vektor Tenzor. Anal. **1** (1933), 102–142.

3. G. Vranceanu, *Leçons de géométrie différentielle*, Vol. 2, Éditions de L'Académie de la République Populaire Roumaine, Bucharest, 1957, pp. 43–51.

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