

# DELETED PRODUCTS WITH PRESCRIBED HOMOTOPY TYPES

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**1. Introduction.** This work is concerned with the problem of deciding when a space  $Y$  has the homotopy type of a deleted product  $RX = X \times X - \text{diagonal}$ . That is, given  $Y$  find spaces  $X$  such that  $RX$  has the homotopy type of  $Y$ . §2 argues that this is a reasonable program by briefly reviewing a small segment of embedding theory. The next two sections present techniques useful in solving the problem, and also note that there are spaces  $Y$  not having the homotopy type of any deleted product. The last two sections apply these techniques to obtain some solutions for the cases  $Y = S^1$  and  $Y = S^2$ .

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**2. Obstructions to embeddings.** Let  $X$  and  $Y$  be Hausdorff spaces. On the *deleted product*

$$RX = \{(x, x') \mid x, x' \in X \text{ and } x \neq x'\}$$

there is the involution  $T_X(x, x') = (x', x)$ . Let  $R_*X = RX/T_X$  and  $E(X, Y) = (RX \times RY)/(T_X \times T_Y)$  be orbit spaces, and let  $\pi: E(X, Y) \rightarrow R_*X$  be the map induced by the projection  $RX \times RY \rightarrow RX$ . Using the fact that  $X$  is Hausdorff, one easily verifies that  $\pi$  is the projection map of a fiber bundle with fiber  $RY$  and group  $Z_2$  (=the integers modulo 2).

An embedding  $f: X \rightarrow Y$  induces a  $Z_2$ -equivariant map  $Rf: RX \rightarrow RY$ , and hence an equivariant map

$$RX \xrightarrow{\text{diagonal}} RX \times RX \xrightarrow{1 \times Rf} RX \times RY.$$

It is immediate that the induced map  $R_*X \rightarrow E(X, Y)$  is a cross section for  $\pi$ . Thus if  $B_n$  is the standard bundle of coefficients  $\pi_n(RY)$  over  $RY$ , the vanishing of the obstructions

$$\gamma^{n+1}(\pi) \in H^{n+1}(R_*X; B_n)$$

for all  $n$  is a necessary condition for the existence of an embedding. These ideas have been studied by van Kampen [6], Whitney [8], Wu [9], Shapiro [5], Haefliger [3] and Weber [7].

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The moral I wish to draw from this is simply that in studying embeddings  $X \rightarrow Y$ , the homotopy types of  $RX$  and  $RY$  are of fundamental importance. Since obstruction theory plays a role here one needs to build universal examples for the obstructions. It is a reasonable conjecture that these examples would be more valuable if they were deleted products themselves. Thus one is confronted with the problem: given a space  $P$ , find  $X$  such that  $RX$  has the homotopy type of  $P$ .

**3. Euler-Poincaré characteristics.** Suppose  $T$  is a finite triangulation of a space  $X$ . Wu [9] shows that

$$JT = U\{\sigma \times \tau \mid \sigma, \tau \in T \text{ and } \sigma \cap \tau = \emptyset\}$$

is a deformation retract of  $RX$  (the simplexes  $\sigma, \tau$  are assumed to be closed). Thus the Euler-Poincaré characteristic  $\chi RX$  is the same as that of  $JT$ . Since the number of cells  $\sigma \times \tau$  in  $JT$  is even, one immediately makes the following observation.

(3.1)  $\chi RX$  is an even integer. Thus there exist spaces which do not have the homotopy type of any deleted product.

Let  $T_p$  be the set of  $p$ -simplexes in the triangulation  $T$ . If  $\sigma, \tau \in T$ , let  $\langle \sigma, \tau \rangle = 0$  when  $\sigma \cap \tau = \emptyset$  and  $\langle \sigma, \tau \rangle = 1$  otherwise. Since a cell  $\sigma \times \tau$  appears in  $JT$  if and only if  $\langle \sigma, \tau \rangle = 0$ ,

$$\begin{aligned} \chi JT &= \sum \{(-1)^{p+q}(1 - \langle \sigma, \tau \rangle) \mid \sigma \in T_p, \\ &\tau \in T_q; p, q = 0, 1, 2, \dots\} \\ (3.2) \quad &= (\chi X)^2 - \sum (-1)^{p+q} \langle \sigma, \tau \rangle. \end{aligned}$$

Now suppose  $\dim X = 1$ . Let  $c_k$  be the number of simplexes in  $T_k$ . If  $\sigma \in T_0$ , the order of  $\sigma$  is

$$O(\sigma) = \sum \{\langle \sigma, \tau \rangle \mid \tau \in T_1\}.$$

Let  $p_n = p_n(X)$  be the number of 0-simplexes of order  $n$ , and abbreviate  $\chi = \chi X$ . Then we have

$$\begin{aligned} \sum \{\langle \sigma, \tau \rangle \mid \sigma, \tau \in T_0\} &= c_0 = \sum p_n, \\ \sum \{\langle \sigma, \tau \rangle \mid \sigma \in T_0, \tau \in T_1\} &= \sum_{\sigma} \left( \sum_{\tau} \langle \sigma, \tau \rangle \right) = \sum n p_n, \\ (3.3) \quad \sum \{\langle \sigma, \tau \rangle \mid \sigma, \tau \in T_1\} &= c_1 + \sum \{\langle \sigma, p \rangle \langle p, \tau \rangle \mid p \in T_0, \sigma \neq \tau\} \\ &= c_1 + \sum_p \left( \sum_{\sigma} \langle p, \sigma \rangle \left( \sum_{\tau \neq \sigma} \langle p, \tau \rangle \right) \right) \\ &= c_1 + \sum n(n-1)p_n. \end{aligned}$$

Using (3.2), (3.3) and the fact that  $\chi = c_0 - c_1$ , we get

$$(3.4) \quad \chi RX = \chi^2 + \chi - \sum (n-2)(n-1)p_n.$$

This yields the topological invariant,

$$\lambda = \lambda(X) = \sum (n-2)(n-1)p_n,$$

which will be used in the last two sections. Variants of (3.4) may be obtained by noting that

$$\sum np_n = \sum_{\tau} \sum_{\sigma} \langle \sigma, \tau \rangle = 2c_1$$

( $\sigma \in T_0$  and  $\tau \in T_1$ ), whence  $2\chi = \sum (2-n)p_n$ .

**4. Suspensions.** Define the *suspension* of a space  $X$  to be the join,

$$SX = X * 2,$$

where  $2 = \{a, b\}$  is a discrete space with exactly two points.

(4.1) THEOREM. *If  $X$  is finitely triangulable, then  $RSX$  has the homotopy type of  $SRX$ .*

PROOF. Let  $T$  be a triangulation of  $X$  and let  $ST$  be the induced triangulation of  $SX$ . Set

$$X_1 = \cup \{(\sigma * a) \times (\tau * b) \mid \sigma, \tau \in T, \sigma \cap \tau = \emptyset\},$$

$$X_2 = \cup \{(\sigma * b) \times (\tau * a) \mid \sigma, \tau \in T, \sigma \cap \tau = \emptyset\}.$$

It is easily checked that  $JST = X_1 \cup X_2$  and  $JT = X_1 \cap X_2$ . The theorem now follows from the observation that  $X_1$  and  $X_2$  are contractible.

The author claims no priority here. This result seems to be known but unpublished.

Note that in this theorem one must use the classical suspension, rather than the reduced suspension. For example, if  $X = k$  is a discrete space with  $k$  points, then  $\chi RSX = \chi SRX = 2 + k - k^2$ . But the reduced suspension  $S'X$  is a wedge of  $k-1$  circles. It follows from (3.4) that  $\chi RS'X = -3(k-1)(k-2)$ .

### 5. Homotopy 1-spheres.

(5.1) THEOREM. *Suppose  $X$  is a 1-dimensional, finitely triangulable space. Then  $RX$  is a homotopy 1-sphere iff*

- (a)  $X$  has no pair of disjoint simple closed curves,
- (b)  $X$  is connected, and
- (c)  $\lambda(X) = \chi^2 + \chi$ .

PROOF. If  $X_1$  and  $X_2$  are disjoint closed subsets of  $X = X_1 \cup X_2$ , then  $RX = RX_1 \cup RX_2 \cup X_1 \times X_2 \cup X_2 \times X_1$ . Furthermore, the only con-

nected finitely triangulable space  $Y$  with  $RY$  disconnected is an arc [4]. Thus  $RX$  is connected iff  $X$  is connected and not an arc. If  $X$  contains a pair of disjoint simple closed curves  $C_1, C_2$ , then  $C_1 \times C_2$  and  $C_2 \times C_1 \subset RX$  carry nontrivial 2-dimensional homology classes.

Now suppose  $RX$  is a homotopy 1-sphere. We have just seen that (a) and (b) must hold. Furthermore,  $\chi RX = \chi(S^1) = 0$ , whence (c) follows from (3.4).

Conversely, suppose (a), (b) and (c) hold. Condition (a) implies that the Betti number  $\beta_2(RX) = 0$  and condition (b) implies  $\beta_0(RX) = 1$  (with (c) used to eliminate the case  $X = \text{an arc}$ ). Condition (c) and (3.4) imply  $\chi RX = 0$ , whence  $\beta_1(RX) = 1$ . C. W. Patty shows when  $X$  is 1-dimensional and finitely triangulable,  $H_*(RX)$  is free abelian and  $\pi_1(RX, *)$  is a freely generated group. Thus  $RX$  is a homotopy 1-sphere.

It is well known that if  $X$  is a 1-sphere or a triod, then  $RX$  is a homotopy 1-sphere. I conjecture that there are no other 1-dimensional spaces with this property.

ADDED IN PROOF. C. W. Patty has pointed out to me that condition (a) in Theorems (5.1) and (6.1) is insufficient to insure that  $RX$  have no 2-cycles. The houses-and-wells configuration and the complete graph on five vertices are counterexamples. In each theorem, the condition should read

(a)  $X$  may be embedded in the plane, and contains no pair of disjoint simple closed curves.

A proof of this should appear soon.

**6. Homotopy 2-spheres.** According to Theorem 4.1, any suspension of one of the spaces constructed in §5 will have its deleted product be a homotopy 2-sphere. Only one of these is contractible. In this section we construct *cones*  $CX$ , with  $X$  a 1-dimensional space, such that  $RCX$  is a homotopy 2-sphere.

(6.1) THEOREM. *Suppose  $X$  is a 1-dimensional, finitely triangulable space. Then  $RCX$  is a homotopy 2-sphere iff*

- (a)  $X$  has no pair of disjoint simple closed curves,
- (b)  $X$  is connected, and
- (c)  $\lambda = \chi^2 - \chi + 2$ .

The proof of Theorem 6.1 is preceded by two lemmas which seem to be of independent interest.

(6.2) LEMMA. *If  $X$  is a finitely triangulable connected space, other than a circle or an arc, then the fundamental group of  $RCX$  is trivial.*

PROOF. Select a triangulation  $T$  for  $X$  and use the induced triangu-

$CT = T*v$  of  $CX = X*v$ , where  $v$  is some point not in  $X$ . It suffices to show that  $\pi_1(JCT)$  is trivial. But  $JCT = X_1 \cup X_2$ , where

$$X_1 = \cup \{(\sigma * v) \times \tau \mid \sigma \cap \tau = \emptyset\},$$

$$X_2 = \cup \{\sigma \times (\tau * v) \mid \sigma \cap \tau = \emptyset\}.$$

Since  $X_1 \cap X_2 = JT$  is connected, it suffices to show that every simple closed curve in  $X_i$  ( $i=1, 2$ ) bounds a disc in  $JCT$ . Since  $v \times X \subset X_1$  and  $X \times v \subset X_2$  are deformation retracts, it suffices to do this for curves  $v \times C$  and  $C \times v$  with  $C$  a simple closed curve in  $X$ . Since  $X$  is not a 1-sphere, there is a point  $p \in X - C$ . Then  $(p*v) \times C \cup p \times (C*v)$  is a disc whose boundary is  $v \times C$ , and  $C \times v$  is the boundary of  $C \times (p*v) \cup (C*v) \times p$ .

(6.3) LEMMA. *If  $X$  is a finitely triangulable 1-dimensional space, then  $\chi RCX = \lambda - \chi^2 + \chi$  ( $\lambda = \lambda(X)$  and  $\chi = \chi(X)$ ).*

PROOF. In the notation of Lemma 6.2,  $JCT = X_1 \cup X_2$  and  $JT = X_1 \cap X_2$ . Thus  $\chi JCT + \chi JT = \chi X_1 + \chi X_2 = 2\chi$ . The result now follows at once from (3.4).

PROOF OF THEOREM (6.1). If  $\beta_k(Y)$  is the  $k$ th Betti number of a space  $Y$ , then

$$\beta_{k+1}(RCX) - \beta_k(RX) + 2\beta_k(X) = 0 \quad \text{for } k > 0;$$

(6.4)

$$\beta_1(RCX) - \beta_0(RX) + 2\beta_0(X) - 1 = 0,$$

provided  $X$  is not a 1-point space, a 2-point space or a 1-sphere [1]. Also,  $RCX$  is connected.

Suppose (a), (b) and (c) are satisfied. Then  $X$  is not a 1-point space, a 2-point space, an arc or a 1-sphere. The fact that  $\dim X = 1$ , together with (6.4) insures that  $\beta_n(RCX) = 0$  for  $n > 3$ . Condition (a) implies  $\beta_2(RX) = 0$ , whence  $\beta_3(RCX) = 0$ . Condition (b) and the remarks above imply  $\beta_0(RX) = \beta_0(X) = 1$ , whence  $\beta_1(RCX) = 0$ . Thus condition (c) and Lemma (6.3) imply  $\beta_2(RCX) = 1$ . That is,  $RCX$  is a homology 2-sphere. Lemma (6.2) now shows that  $RCX$  is a homotopy 2-sphere.

The proof of the converse follows obvious lines.

The triod, the  $\theta$ -curve and a space in the shape of the letter  $P$  (without serifs) all have cones of the same isotopy type, and the deleted product of any one of these is a homotopy 2-sphere. Apart from these, all the spaces satisfying (a), (b), (c) and having distinct topological types have cones of distinct isotopy types [2]. A family

of such spaces is obtained by taking  $n$ -gons and joining their vertices to their centers.

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