SUBDIRECT SUMS OF INTEGERS AND REALS

PAUL F. CONRAD[1]

1. Introduction and statement of the main theorems. The concept of a subdirect sum of integers is important in the study of lattice-ordered groups ("l-groups") since Weinberg [12] has shown that a free abelian l-group is a subdirect sum of integers and hence each abelian l-group is a homomorphic image of a subdirect sum of integers. In this paper those l-groups which are subdirect sums of integers are characterized. We also characterize those l-groups which are subdirect sums of subgroups of the naturally ordered additive group R of real numbers. Topping [10] has shown that each vector lattice is a homomorphic image of such an l-group.

Pappert [9] has determined a necessary and sufficient condition for a vector lattice to be a subdirect sum of reals, and Bernau [2] has shown that with a slight modification her theory applies to an arbitrary l-group. Both of these authors use the fact that an archimedean l-group can be represented by almost finite functions on a Stone space to obtain their results. Our condition is simpler and the proof is elementary.

In [3] Bernau characterizes those subdirect sums of integers which contain the small sum and those which contain a dense subset of bounded elements. We can also characterize these classes of l-groups. These and other special cases and corollaries of our two main theorems are contained in §3.

For each $\lambda \in \Lambda$ let G_λ be a totally ordered group ("o-group") that is o-isomorphic to a subgroup of R. Thus each G_λ is an archimedean o-group or equivalently an o-group without proper convex subgroups. πG_λ will denote the large or unrestricted direct sum of the G_λ ordered pointwise—the *large cardinal sum of the* G_λ—and $\sum G_\lambda$ will denote the *small cardinal sum* of the G_λ. In particular, πG_λ is an l-group and $\sum G_\lambda$ is an l-ideal of πG_λ. If there exists an l-isomorphism of an l-group G onto a subdirect sum of πG_λ, then we say that G is a *subdirect sum of reals*. If, in addition, each G_λ is cyclic, then we say that G is a *subdirect sum of integers*.

Let G be an l-group, $G^+ = \{g \in G \mid g > 0\}$ and let Z^+ be the set of all strictly positive integers. An element $x \in G^+$ will be called *real* if there exists a map $y \rightarrow \bar{y}$ of G^+ into Z^+ such that

Received by the editors May 4, 1967.

[1] This research was supported by a grant from the National Science Foundation.

I. $(\bar{y}x-y)\wedge(\bar{z}x-z)\nleqq 0$ for all $y, z\in G^+$. If, in addition, for all $y\in G^+$ and all $n\in Z^+$,

II. $\bar{y}=1$ implies

$$\overline{ny} = 1,^2$$

III. $x\geqq 2y$ implies $\bar{y}=1$,

then x will be called an *integral* element of G.

THEOREM 1. *An l-group G is a subdirect sum of reals if and only if each $y\in G^+$ exceeds a real element.*

THEOREM 2. *An l-group G is a subdirect sum of integers if and only if each $y\in G^+$ exceeds an integral element.*

2. **Proofs of Theorems 1 and 2.** In all that follows, let $G\neq 0$ be an l-group. A *convex l-subgroup* M of G is a subgroup that satisfies

$$|x| \leqq |a| \quad \text{for } x\in G \text{ and } a\in M \text{ implies } x\in M,$$

or, equivalently, M is a sublattice and a convex subset of G. In particular, the set of all right cosets of a convex l-subgroup M is a distributive lattice such that for all $a, b\in G$

$$M + a \vee M + b = M + a \vee b$$

and dually, where by definition $M+a\geqq M+b$ if $x+a\geqq b$ for some $x\in M$. A *prime* subgroup of G is a convex l-subgroup for which the lattice of right cosets is totally ordered. For a convex l-subgroup M of G the following are equivalent.

(a) M is prime.

(b) The set of convex l-subgroups that contain M is a chain with respect to inclusion.

(c) If $a, b\in G^+\backslash M$, then $a\wedge b\in G^+\backslash M$.

Let $\mathfrak{M}$ be the set of all maximal prime subgroups of G. If $M\in\mathfrak{M}$ and $M\lhd G$, then G/M is o-isomorphic to a subgroup of R (notation $G/M\prec R$). For proofs of the above see [6].

We shall consider the following properties of $x\in G^+$.

(1) There exists $M\in\mathfrak{M}$ such that $M+x$ covers M and for each $y\in G^+$, $M+nx>M+y$ for some $n\in Z^+$.

(2) x is an integral element of G.

(3) x is a real element of G.

(4) There exists $M\in\mathfrak{M}$ such that for each $y\in G^+$, $M+nx>M+y$ for some $n\in Z^+$.

[2] This element will be denoted by $(ny)^-$.

LEMMA. (1)→(2)→(3)↔(4), *and if each* $M \in \mathfrak{M}$ *is normal in* G, *then* (2)→(1).

PROOF. It follows from the definition of real and integral elements that (2)→(3). (4)→(3). For each $y \in G^+$ let $\bar{y}$ be the least element in Z^+ such that $M+\bar{y}x > M+y$. Then for all $y, z \in G^+$

$$M + (\bar{y}x - y) \wedge (\bar{z}x - z) = M + (\bar{y}x - y) \wedge M + (\bar{z}x - z) > M.$$

Thus $(\bar{y}x - y) \wedge (\bar{z}x - z) \nleqq 0$ and so x is real.

(1)→(2). Define $\bar{y}$ as above. Since $M+x$ covers M, for $y \in G^+$ and $n \in Z^+$ the following are equivalent: $\bar{y}=1$, $y \in M$, $ny \in M$ and $(ny)^- = 1$. If $y \in G^+$ and $x \geqq 2y$, then $y \in M$ and so $\bar{y}=1$. For if $y \notin M$, then $M+x \geqq M+2y > M+y > M$, but this contradicts the fact that $M+x$ covers M. Therefore x is an integral element in G.

(3)→(4). For $y, z \in G^+$,

$$[(\bar{y}x - y) \vee 0] \wedge [(\bar{z}x - z) \vee 0] = [(\bar{y}x - y) \wedge (\bar{z}x - z)] \vee 0 \in G^+.$$

Thus $Q_x = \{(\bar{y}x - y) \vee 0 \mid y \in G^+\}$ is contained in an ultrafilter K of G^+. That is, $0 < a \wedge b \in K$ for all $a, b \in K$, and K is maximal with respect to this property. It follows that $N = \bigcup_{k \in K} k'$ is a minimal prime subgroup of G and $K = G^+ \backslash N$, where $k' = \{g \in G \mid |g| \wedge k = 0\}$ is the polar of k. This is Theorem 5.1 in [7], and this result is also implicit in [1] and [8].

(A) $N+\bar{y}x > N+y$ for each $y \in G^+$. For we have $(\bar{y}x - y) \vee 0 \in K = G^+ \backslash N$ and hence $N + (\bar{y}x - y) \vee 0 > N$ and so $N + \bar{y}x - y > N$. Since the convex l-subgroups of G that contain N form a chain, there is a unique convex l-subgroup $M \supseteq N$ that is maximal with respect to $x \notin M$.

(B) $M \in \mathfrak{M}$. For if $y \in G^+$, then $N+\bar{y}x > N+y$ and hence $a+\bar{y}x > y > 0$ for some $a \in N$. But clearly $a+\bar{y}x$ is contained in any convex l-subgroup that properly contains M. Therefore G covers M and hence $M \in \mathfrak{M}$. It follows from (A) that

$$M + (\bar{y} + 1)x > M + \bar{y}x \geqq M + y.$$

Therefore (4) is satisfied.

To complete the proof we need to show that (2)→(1) provided that each $M \in \mathfrak{M}$ is normal in G. Let x be an integral element and let M and N be as above. Suppose (by way of contradiction) that $M+x > M+y > M$ for some $y \in G$. Then since $M+y \vee 0 = M+y \vee M = M+y$ and $M+x \wedge y = M+x \wedge M+y = M+y$, we may assume that $x > y > 0$. Now $x = x - y + y$ and since $x-y, y \in G^+ \backslash M$ and M is prime, $d = (x-y) \wedge y \in G^+ \backslash M$. Clearly $x \geqq 2d$ and hence $\bar{d}=1$ and $(nd)^- = 1$

for all $n\in Z^+$. Thus $M+x=M+(nd)^- x\geqq M+nd\geqq M+d>M$ for all $n\in Z^+$, but this is impossible because $G/M\prec R$.

REMARK. S. Wolfenstein and T. Lloyd have independently shown that if M is a prime subgroup and $M+x$ covers M, then M is normal in the convex l-subgroup that covers it. Thus (1) is equivalent to

(1′) There exists $M\in\mathfrak{M}$ such that $M+x$ covers M.

COROLLARY. *Suppose that each $M\in\mathfrak{M}$ is normal in G and consider $x\in G^+$.*

(a) *x is a real element of G if and only if $x\in G\backslash M$ for some $M\in\mathfrak{M}$.*

(b) *x is an integral element of G if and only if $M+x$ covers M for some $M\in\mathfrak{M}$.*

PROOF. This is an immediate consequence of the lemma and the fact that $G/M\prec R$ is an archimedean o-group for each $M\in\mathfrak{M}$.

Byrd [4] has shown that G is a subdirect sum of o-groups if and only if for each prime subgroup M and each $g\in G$, $-g+M+g\subseteq M$ or $-g+M+g\supseteq M$. Thus for this class of l-groups each $M\in\mathfrak{M}$ is normal.

PROOF OF THEOREM 1. Suppose that G is a sublattice and a subdirect sum of πR_λ $(\lambda\in\Lambda)$, where each $R_\lambda\subseteq R$. If $x\in G^+$, then $x_\lambda>0$ for some $\lambda\in\Lambda$. Let $M=\{g\in G \mid g_\lambda=0\}$. Then $M\in\mathfrak{M}$ and $x\in G\backslash M$. Thus by the corollary, x is real, and so *each $x\in G^+$ is real.*

Conversely suppose that each element in G^+ exceeds a real element, and consider y, $z\in G^+$. There exists a real element $x\leqq z$. Thus $\bar{y}x\nleqq y$ and hence $\bar{y}z\nleqq y$. Therefore G is archimedean and hence abelian. By the corollary, $x\in G\backslash M$ for some $M\in\mathfrak{M}$ and hence $z\in G\backslash M$. Therefore $0=\bigcap\{M \mid M\in\mathfrak{M}\}$ and so G is a subdirect sum of reals.

PROOF OF THEOREM 2. Suppose that G is a sublattice and a subdirect sum of πZ_λ $(\lambda\in\Lambda)$, where each $Z_\lambda=Z$. If $g\in G^+$, then $g\geqq x>0$ for some $x\in G$ where $x_\lambda=1$ for some $\lambda\in\Lambda$. Let $M=\{g\in G \mid g_\lambda=0\}$. Then $M\in\mathfrak{M}$ and $M+x$ covers M, and hence by the corollary x is integral. Therefore each element in G^+ exceeds an integral element.

Conversely, suppose that each element in G^+ exceeds an integral element. Then, as in the proof of Theorem 1, G is abelian. Let $\mathfrak{I}=\{M\in\mathfrak{M} \mid G/M$ is cyclic$\}$. Then by the corollary $\bigcap\{M \mid M\in\mathfrak{I}\}$ must be zero since it contains no integral element. Therefore G is a subdirect sum of integers.

3. Special cases of Theorems 1 and 2.

An element $s\in G^+$ is called *basic* if $\{g\in G \mid 0\leqq g\leqq s\}$ is totally ordered.

PROPOSITION A. *For an l-group G the following are equivalent.*

(1) *G is a subdirect sum of reals that contains the small sum.*

(2) *Each element in G^+ exceeds a real element that is also basic.*

(3) *G is archimedean and each element in G^+ exceeds a basic element.*

PROOF. It is shown in [5] that (1)↔(3). If each element in G^+ exceeds a real element, then G is archimedean and hence (2)→(3). If (1) holds, then each element in G^+ is real, and hence (1) and (3) imply (2).

There are many other equivalent conditions proven in the literature—see for example [11].

An element $a \in G^+$ is an *atom* if it covers 0. It is shown in [5] that x is a basic element in an archimedean l-group G if and only if $x'' \prec R$ and G is the cardinal sum of x'' and x'. Thus a basic element x is integral if and only if x'' is cyclic, and hence if and only if x is an atom.

PROPOSITION B. *For an l-group G the following are equivalent.*

(1) *G is a subdirect sum of integers that contains the small sum.*

(2) *Each element in G^+ exceeds an integral element that is also basic.*

(3) *G is archimedean and each element in G^+ exceeds an atom.*

PROOF. Clearly (1)→(2)→(3).

(3)→(1). Since each atom is a basic element it follows from Proposition A that G is a subdirect sum of reals that contains the small sum. Thus without loss of generality

$$\sum R_\lambda \subseteq G \subseteq \pi R_\lambda,$$

where $R_\lambda \subseteq R$ for each $\lambda \in \Lambda$. If R_λ is not cyclic, then there exists an element in $R_\lambda^+ \subseteq G^+$ that does not exceed an atom, a contradiction. Therefore (1) holds.

An element $s \in G^+$ is called *singular* if $a \wedge (s-a) = 0$ for each $0 \leqq a \leqq s$.

PROPOSITION C. *For an l-group G the following are equivalent.*

(1) *G is a subdirect sum of integers and each element in G^+ exceeds a bounded element.*

(2) *Each element in G^+ exceeds an integral element that is also singular.*

(3) *G is a subdirect sum of reals and each element in G^+ exceeds a singular element.*

PROOF. In [7] it is shown that (1)↔(3) and clearly (2)→(3). Suppose that (1) and (3) hold. Then without loss of generality $G \subseteq \pi Z_\lambda$, where for each $\lambda \in \Lambda$, $Z_\lambda = Z$. In [7] it is shown that if $s \in G$ is singular, then $s_\lambda = 1$ or 0. Thus it follows that s is integral and hence we have (2).

Bernau [3] has established (1)↔(3) in Proposition B and has derived a condition that is equivalent to (1) in Proposition C.

Suppose that $x \in G^+$ is real and let A_x be the set of all maps $\pi: G^+ \to Z^+$ such that for all $y, z \in G^+$

$$((y\pi)x - y) \wedge ((z\pi)x - z) \nleq 0.$$

For $\alpha, \beta \in A_x$, define $\alpha \leqq \beta$ if $y\alpha \leqq y\beta$ for all $y \in G^+$. Then $(A_x, \leqq)$ is a po-set and each element in A_x exceeds a minimal element in A_x. For if $\{\alpha_\lambda \mid \lambda \in \Lambda\}$ is a chain in A_x, then for each $y \in G^+$ define

$$y\pi = \min\{y\alpha_\lambda \mid \lambda \in \Lambda\}.$$

If $y, z \in G^+$, then there exists $\lambda \in \Lambda$ such that $y\alpha_\lambda$ and $z\alpha_\lambda$ are minimal and so

$$((y\pi)x - y) \wedge ((z\pi) - z) = ((y\alpha_\lambda)x - y) \wedge ((z\alpha_\lambda)x - z) \nleq 0.$$

Therefore $\pi \in A_x$, and hence by Zorn's lemma each map in A_x exceeds a minimal map.

Definition. A real element $x \in G^+$ for which there exists a minimal map $y \to \bar{y}$ in A_x that also satisfies (II) will be called a ***-element**.

Proposition D. *For an l-group the following are equivalent.*

(1) *Each element in G^+ exceeds a *-element.*

(2) *G is (l-isomorphic to) a subdirect sum of πZ_λ, where for each $\lambda \in \Lambda$, $Z_\lambda = Z$; and $G_\lambda = \{g \in G \mid g_\lambda = 0\}$ is both a maximal and a minimal prime subgroup of G.*

Proof. (1)→(2). Since each *-element is real, it follows from Theorem 1 that G is abelian. Let x be a *-element in G and let $y \to \bar{y}$ be a minimal map in A_x that also satisfies (II). Construct M and N as in the proof of (3)→(4) in the lemma. Since $N + \bar{y}x > N + y$ for all $y \in G^+$ and the map $y \to \bar{y}$ is minimal, it follows that $\bar{y}$ is the least element in Z^+ for which $N + \bar{y}x > N + y$. Suppose (by way of contradiction) that $M \supset N$ and pick $0 < z \in M \backslash N$ and let $y = -(x \wedge z) + x$. Then

$$M + x = M + y \quad \text{and} \quad N + x > N + y.$$

Therefore $\bar{y} = 1$ and hence $\overline{(2y)} = 1$, but clearly $N + \overline{(2y)}x = N + x < N + 2y$, a contradiction. Thus $N = M$ is both maximal and minimal. If $M + x > M + y$, then $\bar{y} = 1$ and hence $M + x = M + \overline{(ny)}x \geqq M + ny$ for all $n \in Z^+$. Thus since $G/M \prec R$ it follows that $y \in M$ and so G/M is cyclic.

(2)→(1). We may assume that $G \subseteq \pi Z_\lambda$. If $z \in G^+$, then $z \geqq x \in G^+$, where $x_\lambda = 1$ for some $\lambda \in \Lambda$. For $y \in G^+$ define $\bar{y}$ to be the least element in Z^+ such that $\bar{y}x_\lambda > y_\lambda$. Then the map $y \to \bar{y}$ satisfies (I), (II) and (III). It remains to be shown that this map is minimal in A_x. Suppose that $\pi \in A_x$ and that $y\pi \leqq \bar{y}$ for all $y \in G^+$. Construct M and N as above using the map $y \to y\pi$. In particular, $N+(y\pi)x > N+y$ and $M+(y\pi)x \geqq M+y$ for all $y \in G^+$.

If $M \neq G_\lambda$, then there exists $y \in G^+$ such that $y_\lambda = 0$ and $y \notin M$. Since $y_\lambda = 0$, $\bar{y} = 1$ and so $(ny)^- = (ny)\pi = 1$ for all $n \in Z^+$, but this means that $M+x = M+((ny)\pi)x \geqq M+ny$ for all $n \in Z^+$, and this contradicts the fact that $G/M \prec R$.

If $M = G_\lambda$, then since G_λ is a minimal prime $M = N$ and so $M+(y\pi)x > M+y$ for all $y \in G^+$, and it follows that $\bar{y} = y\pi$ for all $y \in G^+$. Therefore x is a *-element and so (1) is satisfied.

References

1. B. Banaschewski, *On lattice ordered groups*, Fund. Math **55** (1964), 113–123.

2. S. Bernau, *Unique representations of archimedean lattice groups and normal archimedean lattice rings*, Proc. London Math. Soc. **15** (1965), 599–631.

3. ———, *On subdirect sums of integers* (preprint).

4. R. Byrd, *Complete distributivity in lattice ordered groups*, Pacific J. Math. **20** (1967), 423–432.

5. P. Conrad, *Some structure theorems for lattice-ordered groups*, Trans. Amer. Math. Soc. **99** (1961), 1–29.

6. ———, *The lattice of all convex l-subgroups of a lattice-ordered group*, Czechoslovak Math. J. **15** (1965), 101–123.

7. P. Conrad and D. McAlister, *The completion of a lattice ordered group*, J. Austral. Math. Soc. (to appear).

8. D. Johnson and J. Kist, *Prime ideals in vector lattices*, Canad. J. Math. **14** (1962), 517–528.

9. D. Papert, *A representation theory for lattice groups*, Proc. London. Math. Soc. **12** (1962), 100–120.

10. D. Topping, *Some homological pathology in vector lattices*, Canad. J. Math. **17** (1965), 411–428.

11. E. Weinberg, *Completely distributive lattice-ordered groups*, Pacific J. Math. **12** (1962), 1131–1137.

12. ———, *Free lattice ordered abelian groups*, Math. Ann. **159** (1965), 217–222.

Tulane University