ON AN ANALYTIC SIMPLIFICATION OF A SYSTEM OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS CONTAINING A PARAMETER[1]

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1. **Introduction.** Let x be a complex variable and let D_0 be a simply connected compact domain in the x-plane which contains the origin O in its interior. Let ρ_0 be a positive number. Consider a system of linear ordinary differential equations of the form

$$\epsilon^\sigma dy/dx = A(x, \epsilon)y \tag{1.1}$$

where σ is a nonnegative integer, ϵ is a complex parameter, y is an n-dimensional vector, and $A(x, \epsilon)$ is an n by n matrix with components holomorphic in a domain

$$x \in D_0, \qquad |\epsilon| \leqq \rho_0. \tag{1.2}$$

Let

$$A(x, \epsilon) = \sum_{k=0}^{\infty} \epsilon^k A_k(x) \tag{1.3}$$

be the expansion of $A(x, \epsilon)$ in powers of ϵ, where $A_k(x)$ are holomorphic in D_0.

In this paper, we shall prove the following theorem.

THEOREM. *For each nonnegative integer m, there exists an n by n matrix $P(x, \epsilon)$ satisfying the following conditions:*

(i) *the components of $P(x, \epsilon)$ are holomorphic with respect to (x, ϵ) in the domain*

$$x \in D_1, \qquad |\epsilon| \leqq \rho_0, \tag{1.2$'$}$$

where D_1 is a certain subdomain of D_0 which contains the origin O in its interior;

(ii) $P(x, 0) = 1_n$ *for* $x \in D_1$ *and* $P(0, \epsilon) = 1_n$ *for* $|\epsilon| \leqq \rho_0$, *where* 1_n *is the n by n unit-matrix;*

(iii) *the system* (1.1) *is reduced to*

$$\epsilon^\sigma du/dx = \left\{ \sum_{k=0}^{m} \epsilon^k A_k(x) + \epsilon^{m+1} \sum_{k=0}^{\sigma-1} \epsilon^k B_k(x) \right\} u \tag{1.4}$$

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by the transformation

$$y = P(x, \epsilon)u \tag{1.5}$$

where $B_k(x)$ $(k=0, 1, \cdots, \sigma-1)$ *are* n *by* n *matrices whose components are holomorphic for* $x \in D_1$.

REMARK. In case when $\sigma=1$, the right-hand member of (1.4) has the form

$$\left\{ \sum_{k=0}^{m} \epsilon^k A_k(x) + \epsilon^{m+1} B_0(x) \right\} u.$$

In particular if $\sigma=1$ and $m=0$, the system (1.4) has the form

$$\epsilon du/dx = \{A_0(x) + \epsilon B_0(x)\} u.$$

G. D. Birkhoff [2] has proved a result similar to ours for linear differential equations at an irregular singular point. Since his result was concerned with the behavior of solutions at a singular point with respect to the independent variable, it was necessary to assume a certain condition on the monodromy matrix at the singular point. (See, for example, H. L. Turrittin [5].) We do not need to assume such a condition, insomuch as our result is only concerned with the singularity with respect to the parameter.

It might be possible to prove our theorem by using a method similar to that of Birkhoff's result. However, it is necessary to modify his lemma on matrices [1] in such a manner that this lemma can be used for matrices depending on many variables. Instead of doing this, we shall prove our theorem by using a direct method which is based on the theory of ordinary differential equations in a Banach space. This method was suggested by Y. Sibuya in one of his papers [4]. The author is indebted to Professor Yasutaka Sibuya for valuable discussions during this work.

2. **Fundamental nonlinear equations.** Let us put

$$P(x, \epsilon) = 1_n + \epsilon^{m+1} \sum_{k=0}^{\infty} \epsilon^k P_k(x) \tag{2.1}$$

and

$$B(x, \epsilon) = \sum_{k=0}^{m+\sigma} \epsilon^k B_k(x), \tag{2.2}$$

where

$$(2.3)\qquad \begin{aligned} \hat{B}_k(x) &= A_k(x), \qquad (k = 0, 1, \cdots, m), \\ &= B_{k-m-1}(x), \qquad (k = m+1, m+2, \cdots, m+\sigma). \end{aligned}$$

In order that the transformation (1.5) reduces the system (1.1) to the system (1.4), we must have the differential equation

$$(2.4)\qquad \epsilon^\sigma dP/dx = A(x, \epsilon)P - PB$$

satisfied by the matrices P and B. From this equation we derive

$$(2.5)\qquad \begin{aligned} 0 = A_{m+1+k}(x) &- \hat{B}_{m+1+k}(x) \\ &+ \sum_{h=0}^{k} \{A_{k-h}(x)P_h(x) - P_h(x)\hat{B}_{k-h}(x)\}, \\ &\qquad (k = 0, 1, \cdots, \sigma - 1) \end{aligned}$$

and

$$(2.6)\qquad \begin{aligned} dP_k(x)/dx = A_{m+1+\sigma+k}(x) &+ \sum_{h=0}^{\sigma+k} A_{\sigma+k-h}(x)P_h(x) \\ &- \sum_{h=k-m}^{\sigma+k} P_h(x)\hat{B}_{\sigma+k-h}(x), \qquad (k = 0, 1, 2, \cdots), \end{aligned}$$

where

$$(2.7)\qquad P_h(x) \equiv 0 \quad \text{if} \quad h < 0.$$

We shall determine P and B by solving these equations.

We should remark here that, in many cases, formal power series P and B which satisfy the equation (2.4) are not convergent. (See, for example, Y. Sibuya [3] and W. Wasow [6].) In order to get P as a convergent power series in ϵ we must choose a suitable B. To do this, first of all, let us solve (2.5) with respect to $\hat{B}_{m+1+k}(x)$. Then we get

$$(2.8)\qquad \begin{aligned} \hat{B}_{m+1+k}(x) = A_{m+1+k}(x) &+ H_{m+1+k}(x; P_0, \cdots, P_k), \\ &\qquad (k = 0, 1, 2, \cdots, \sigma - 1) \end{aligned}$$

where H_j are defined by

$$H_j = 0, \qquad (j = 0, 1, \cdots, m),$$

$$(2.9)\qquad \begin{aligned} H_{m+1+k}(x; P_0, \cdots, P_k) &= \sum_{h=0}^{k} \{A_{k-h}(x)P_h - P_hA_{k-h}(x)\} \\ &\quad - \sum_{h=0}^{k} P_hH_{k-h}, \\ &\qquad (k = 0, 1, \cdots, \sigma - 1). \end{aligned}$$

Substituting (2.8) into (2.6) we get

$$dP_k/dx = f_k(x; \mathfrak{P}), \qquad (k = 0, 1, 2, \cdots), \tag{2.10}$$

where

$$\begin{aligned} f_k(x; \mathfrak{P}) = A_{m+1+\sigma+k}(x) + \sum_{h=0}^{\sigma+k} A_{\sigma+k-h}(x)P_h \\ - \sum_{h=k-m}^{\sigma+k} P_h A_{\sigma+k-h}(x) - \sum_{h=k-m}^{\sigma+k} P_h H_{\sigma+k-h}(x; \mathfrak{P}), \\ (k = 0, 1, 2, \cdots), \end{aligned} \tag{2.11}$$

with $\mathfrak{P}$ denoting an infinite-dimensional vector $\{P_k;\ k=0, 1, \cdots\}$. If we denote by $\mathfrak{f}(x; \mathfrak{P})$ the infinite-dimensional vector

$$\{f_k(x; \mathfrak{P}); \qquad k = 0, 1, 2, \cdots\},$$

then equations (2.11) can be written in the form

$$d\mathfrak{P}/dx = \mathfrak{f}(x; \mathfrak{P}). \tag{2.12}$$

We shall solve this differential equation in a suitable Banach space. If $\mathfrak{P}$ is determined, then the quantities B_k are determined by (2.8) and (2.9).

3. **A lemma on** $\mathfrak{f}(x; \mathfrak{P})$. Since components of the matrix $A(x, \epsilon)$ are holomorphic in the domain (1.2), there is a positive number ρ such that $\rho > \rho_0$ and that components of A are holomorphic in the domain

$$x \in D_0, \qquad |\epsilon| \leqq \rho. \tag{3.1}$$

Let us denote by $\mathfrak{B}$ the set of all infinite-dimensional vectors $\mathfrak{P} = \{P_k;\ k=0, 1, 2, \cdots\}$ such that

(i) P_k are n by n matrices whose components are complex numbers;

(ii) $\sum_{k=0}^{\infty} \rho^k |P_k| < +\infty$,

where $|P_k|$ is the sum of absolute values of components of P_k. For each $\mathfrak{P}$, let us define a norm $\|\mathfrak{P}\|$ by

$$\|\mathfrak{P}\| = \sum_{k=0}^{\infty} \rho^k |P_k|. \tag{3.2}$$

Then we can regard $\mathfrak{B}$ as a Banach space over the field of complex numbers.

Let $\mathfrak{P}(x)$ be a mapping from D_0 to $\mathfrak{B}$. This mapping is said to be $\mathfrak{B}$-holomorphic in D_0 if there exists another mapping $\mathfrak{Q}(x)$ from D_0 to $\mathfrak{B}$ such that

(3.3) $$\lim_{h\to 0} \left\| h^{-1}\{\mathfrak{P}(x+h) - \mathfrak{P}(x)\} - \mathfrak{Q}(x)\right\| = 0$$

for all $x \in D_0$. We denote $\mathfrak{Q}$ by $d\mathfrak{P}/dx$. If $\mathfrak{P}(x) = \{P_k(x); k = 0, 1, 2, \cdots\}$ is $\mathfrak{B}$-holomorphic in D_0, then each matrix $P_k(x)$ is holomorphic in D_0 and

(3.4) $$d\mathfrak{P}(x)/dx = \{dP_k(x)/dx; k = 0, 1, 2, \cdots\}.$$

Now we can prove the following lemma.

LEMMA. *Let* $\mathfrak{f}(x, \mathfrak{P})$ *be the infinite-dimensional vector whose components* $f_k(x; \mathfrak{P})$ *are given by* (2.11). *Then* $\mathfrak{f}(x; \mathfrak{P})$ *is a mapping from* $D_0 \times \mathfrak{B}$ *to* $\mathfrak{B}$ *which has the following properties*:

(i) *for each positive number* R *there are two positive numbers* $G(R)$ *and* $K(R)$ *such that*

(3.5) $$\|\mathfrak{f}(x; \mathfrak{P})\| \leqq G(R) \qquad \textit{for } \|\mathfrak{P}\| \leqq R$$

and

(3.6) $$\|\mathfrak{f}(x; \mathfrak{P}) - \mathfrak{f}(x; \tilde{\mathfrak{P}})\| \leqq K(R)\|\mathfrak{P} - \tilde{\mathfrak{P}}\| \qquad \textit{for } \|\mathfrak{P}\| \leqq R, \|\tilde{\mathfrak{P}}\| \leqq R;$$

(ii) $\mathfrak{f}(x; \mathfrak{P}(x))$ *is* $\mathfrak{B}$-*holomorphic in* D_0 *if* $\mathfrak{P}(x)$ *is* $\mathfrak{B}$-*holomorphic in* D_0.

Let us consider a formal power series in ϵ which is defined by

(3.6) $$F(x, \mathfrak{P}, \epsilon) = \sum_{k=0}^{\infty} \epsilon^k f_k(x; \mathfrak{P}).$$

Then from the definition of f_k we derive the following formal identity:

(3.7) $$\begin{aligned} F(x, \mathfrak{P}, \epsilon) = \sum_{k=0}^{\infty} \epsilon^k A_{m+1+\sigma+k}(x) &+ \frac{1}{\epsilon^\sigma}\left\{ \sum_{k=0}^{\infty} \epsilon^k A_k(x) \sum_{k=0}^{\infty} \epsilon^k P_k \right. \\ &\left. - \sum_{k=0}^{\sigma-1} \epsilon^k \sum_{h=0}^{k} A_{k-h}(x) P_h \right\} \\ &- \frac{1}{\epsilon^\sigma}\left\{ \sum_{k=0}^{\infty} \epsilon^k P_k \sum_{k=0}^{m+\sigma} \epsilon^k [A_k(x) + H_k(x; \mathfrak{P})] \right. \\ &\left. - \sum_{k=0}^{\sigma-1} \epsilon^k \sum_{h=0}^{k} P_h [A_{k-h}(x) + H_{k-h}(x; \mathfrak{P})] \right\}. \end{aligned}$$

By using (3.7), we can prove the Lemma in a straightforward manner.

4. **Proof of Theorem.** We shall construct the matrix $P(x, \epsilon)$ by solving the differential equation (2.12) with the initial condition

$$\mathfrak{P}(0) = 0. \tag{4.1}$$

To do this, we use the method of successive approximations. By virtue of the Lemma of §3, we can construct, in this manner, a unique solution $\mathfrak{P}(x)$ which is $\mathfrak{B}$-holomorphic in a subdomain D_1 of D_0 which contains 0 in its interior. Since $d\mathfrak{P}(x)/dx$ is given by (3.4), the solution $\mathfrak{P}(x)$ gives the desired matrix $P(x, \epsilon)$. The matrix $B(x, \epsilon)$ is determined by (2.8) and (2.9). This completes the proof of our theorem.

References

1. G. D. Birkhoff, "A theorem on matrices of analytic functions" pp. 240–251 in *Collected mathematical papers*, Vol. I, Amer. Math. Soc., Providence, R. I., 1950.

2. ———, "Equivalent singular points of ordinary differential equations," pp. 252–257 in *Collected mathematical papers*, Vol. I, Amer. Math. Soc., Providence, R.I., 1950.

3. Y. Sibuya, *Sur réduction analytique d'un système d'équations différentielles ordinaires linéaires contenant un paramètre*, J. Fac. Sci. Univ. Tokyo Sect. I **7** (1958), 527–540.

4. ———, *On the convergence of formal solutions of systems of linear ordinary differential equations containing a parameter*, MRC Tech. Sum. Report #511, September, 1964.

5. H. L. Turrittin, *Reduction of ordinary differential equations to the Birkhoff canonical form*, Trans. Amer. Math. Soc. **107** (1963), 485–507.

6. W. Wasow, *Asymptotic expansions for ordinary differential equations*, Wiley, New York, 1965.

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