PROPERTIES OF BOUNDED SOLUTIONS OF NONLINEAR EQUATIONS OF SECOND ORDER

A. G. KARTSATOS

In this paper we are concerned with the properties of the bounded solutions of differential equations having the form

$$\text{(E)} \qquad \ddot{x} + f(t)g(x, \dot{x}) = 0.$$

Here we consider only solutions of (E) which are defined on some ray $[c, +\infty)$, $c \geqq 0$ (depending on the particular solution), and their existence will be assumed without further mention.

An oscillatory solution $x(t)$, $t \in [c, +\infty)$ of (E), is (by definition) a solution such that for any $t > c$, there exists a $t_1 > t$ with $x(t_1) = 0$.

In the first section we give a theorem in which $f(t)$ is allowed to be negative part of the time, and in the second section we give a criterion in order that all bounded solutions of (E) oscillate.

1. We prove the following

THEOREM 1. *Consider* (E) *under the assumptions*:

(i) $f: I \to \boldsymbol{R} = (-\infty, +\infty)$, $I = [t_0, +\infty)$, $t_0 \geqq 0$, *continuous on* I, *and such that*

$$\int_{t_0}^{\infty} t[\mu f_+(t) + f_-(t)]dt = +\infty, \qquad \textit{for every } \mu > 0,$$

where $f_+(t) = \max\{f(t), 0\}$, *and* $f_-(t) = \min\{f(t), 0\}$;

(ii) g *is defined and continuous on* $\boldsymbol{R}^2$, $xg(x, y) > 0$ *for every* $(x, y) \in (\boldsymbol{R} \setminus \{0\}) \times \boldsymbol{R}$, *and such that: to every pair of constants* l, m *with* $0 < l < m$ *there corresponds a pair of constants* $L = L(l, m)$, $M = M(l, m)$ *with* $0 < L < |g(x, y)| < M$ *for every* (x, y) *with* $l < |x| < m$; *then, if* $x(t)$ *is a bounded solution of* (E), *it must be oscillatory or such that*

$$\liminf_{t \to +\infty} |x(t)| = 0.$$

PROOF. Suppose that there exists a bounded nonoscillatory solution $x(t)$, $t \in [t_1, +\infty)$, $t_1 \geqq t_0$. Then, without any loss of generality, we assume that $x(t) > 0$, $t \in [t_1, +\infty)$. If $\liminf_{t \to +\infty} x(t) > 0$, then, according to (ii), there exists $T \geqq t_1$ such that $\alpha < x(t) < \beta$, and $K < g(x, \dot{x}) < L$ for every $t \in [T, +\infty)$, where α, β are two positive constants and K, L are also positive constants depending on α, β.

Received by the editors September 28, 1966 and, in revised form, March 21, 1967.

Now consider the function $F(t) = t\dot{x}(t)$, $t \in [T, +\infty)$; by differentiation of F we obtain

$$\dot{F}(t) = \dot{x}(t) - tf(t)g(x(t), \dot{x}(t)) \tag{1}$$

which by integration from T to t ($t \geqq T$) gives

$$F(t) = F(T) + x(t) - x(T) - \int_T^t sf(s)g(x(s), \dot{x}(s))ds. \tag{2}$$

Thus, from (2), because of the boundedness of $x(t)$ and $g(x(t), \dot{x}(t))$, we get

$$\begin{aligned} F(t) &\leqq F(T) + \alpha - \beta - \int_T^t sf_+(s)g(x(s), \dot{x}(s))ds \\ &\quad - \int_T^t sf_-(s)g(x(s), \dot{x}(s))ds \\ &\leqq F(T) + \alpha - \beta - L\int_T^t s[(K/L)f_+(s) + f_-(s)]ds. \end{aligned} \tag{3}$$

From (3) we obtain a contradiction, for it yields

$$\lim_{t \to +\infty} F(t) = -\infty, \tag{4}$$

i.e., there exists a constant $M > 0$ such that

$$\dot{x}(t) < -M/t, \qquad t \in [T_1, +\infty) \tag{5}$$

for some $T_1 \geqq T$, which implies $\lim_{t \to +\infty} x(t) = -\infty$. Since we have supposed that $x(t) > 0$, $t \in [t_0, +\infty)$, the contradiction follows. Thus, our assertion is true.

2. We establish

THEOREM 2. *Let the equation* (E) *be such that*:

(i) f *is defined and continuous on the interval* $I = [t_0, +\infty)$, $t_0 \geqq 0$, *positive and such that*

$$\int_{t_0}^{+\infty} tf(t)dt = +\infty;$$

(ii) g *is defined and continuous on* $\mathbf{R}^2$, *and* $xg(x, y) > 0$ *for every* $x \neq 0$; *then every bounded solution of* (E) *is oscillatory.*

PROOF. Assume that there exists a bounded solution $x(t)$ of (E) which is positive on $[t_1, +\infty)$, $t_1 \geqq t_0$; then it is easy to see (by use of the fact that $\ddot{x}(t)<0$) that the derivative $\dot{x}(t)$ is a positive decreasing function on $[t_1, +\infty)$, so that $x(t)$ is increasing on the same interval. Moreover, since $x(t)$ is bounded, we must have $\lim_{t\to+\infty}\dot{x}(t)=0$. Now we find a lower bound for the function $g(x(t), \dot{x}(t))$. Let λ be the limit of $x(t)$ as t tends to infinity $(0<\lambda<+\infty)$; then if ϵ is a fixed constant less than $g(\lambda, 0)$, there exists a $t_2 \geqq t_1$ such that

$$g(\lambda, 0) - \epsilon < g(x(t), \dot{x}(t)) < g(\lambda, 0) + \epsilon \tag{6}$$

for every $t \geqq t_2$.

Thus, as in Theorem 1, we have

$$\begin{aligned} t\dot{x}(t) &\leqq k - \int_{t_2}^{t} sf(s)g(x(s), \dot{x}(s))ds \\ &\leqq k - (g(\lambda, 0) - \epsilon)\int_{t_2}^{t} sf(s)ds \end{aligned} \tag{7}$$

where $k=t_2\dot{x}(t_2)-x(t_2)+\lambda$. From (7) we obtain $\lim_{t\to+\infty} t\dot{x}(t)=-\infty$, contradicting the positivity of $\dot{x}(t)$. A similar argument can be used in the case of an eventually negative solution. Thus, the proof is complete.

REMARK 1. An example of a function satisfying (ii) of Theorem 1 is the following: $g(x, y)=x^3(1+|y|/(1+|y|))$.

REMARK 2. It is possible that the assumptions of Theorem 1 imply that all bounded solutions of (E) are oscillatory, but we are unable to prove it.

REMARK 3. As a consequence of Theorem 2 we obtain the interesting result that all solutions of the equation

$$\ddot{x} + p(t)x = 0, \quad \left(\int_{t_0}^{+\infty} tp(t)dt = +\infty\right) \tag{*}$$

with $0<p(t)\leqq 1/4t^2$, are unbounded, because it is well known that in this case all solutions of (*) are nonoscillatory.

REMARK 4. Theorem 2 improves the sufficiency part of a result of Wong in [1], who considered a special case of the function $g(x, y)$, and showed that the integral condition in (i) of Theorem 2, is necessary and sufficient for all bounded solutions to oscillate.

REFERENCE

1. J. S. Wong, *Some properties of solutions of* $u''+a(t)f(u)g(u')=0$. III, SIAM J. Appl. Math. **14** (1966), 209–214.

WAYNE STATE UNIVERSITY AND
UNIVERSITY OF ATHENS