NORM-COMPACT SETS OF REPRESENTING MEASURES

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Let K be a compact subset of the complex plane $\mathbf{C}$ and let p be a point of K^0, the interior of K. Let $R(K)$ be the uniform closure on K of the rational functions with poles off K and let M_p be the set of all positive measures λ on ∂K, the topological boundary of K, with the property that $\int_{\partial K}\Phi d\lambda = \Phi(p)$ for all Φ in $R(K)$. (Such a measure is called a representing measure for evaluation at p.) In an effort to cast some light on the problem of putting analytic structure in the maximal ideal space of a function algebra, Bishop has conjectured in [**1**, problem 8, p. 347] that M_p is compact in the norm topology as a subset of the space measures on ∂K. Theorem 1 of this paper shows that this is indeed the case for many compact sets; however, Theorem 2 gives some necessary conditions that M_p be norm-compact and consequently provides a number of counterexamples to Bishop's conjecture.

If the compact set has only a finite number of components in its complement, then M_p is norm-compact. This follows immediately from the fact that the linear span of the real measures that annihilate $R(K)$ is finite-dimensional. (This is a consequence of a classical theorem of Walsh [**6**, p. 518].) However, when K has infinitely many complementary components this argument is no longer valid, for both M_p and the space of real annihilating measures may contain infinitely many linearly independent elements. For example, let K be the set obtained by deleting from the closed unit disc a sequence $\{C_i\}$ of open subdiscs with disjoint closures whose centers lie on the positive real axis and increase to 1 and whose radii decrease to 0. Let μ be harmonic measure on ∂K for p and let f_i be the element of $L^\infty(\partial K, \mu)$ such that for each g in $L^1(\partial K, \mu)$, $\int_{\partial K} g f_i d\mu$ is the period about C_i of the harmonic conjugate of the harmonic extension of g to K^0. It is easily seen that $f_1, f_2, \cdots$ are linearly independent and consequently that the representing measures $\lambda_i = (1-(f_i/c_i))\mu$ are linearly independent, where $c_i = \|f_i\|_\infty$. Nevertheless, M_p is norm-compact for this and other compact sets. In order to prove this we must make an assumption about two subsets of ∂K; these two subsets are defined below.

Let K be compact and let $C_0, C_1, \cdots$ be the components of $\mathbf{C}-K$. For each integer n, $n=0, 1, 2, \cdots$, let F_n consist of those points of

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∂C_n which lie in ∂C_i for some $i \neq n$ or are limit points of such points; that is,

$$F_n = \partial C_n \cap \overline{\partial K - \partial C_n};$$

let $F = \bigcup_{n=0}^{\infty} F_n$. F is one of the subsets of interest. The other is the set E of all points in ∂K which do not lie in ∂C_n for any n; that is, $E = \partial K - \bigcup_{n=0}^{\infty} \partial C_n$.

Fundamental to the proof of Theorem 1 is this lemma.

LEMMA 1. *Let K be a compact subset of* **C** *with complementary components $C_0, C_1, \cdots$, and let E and F be the subsets of ∂K defined above. Suppose that $\lambda(E \cup F) = 0$ for all λ in M_p. Then given $\epsilon > 0$, there is an integer N depending only on ϵ such that for each λ in M_p, the total variation of λ over $\partial C_0 \cup \cdots \cup \partial C_N$ exceeds $1 - \epsilon$.*

PROOF. M_p is a subset of the unit sphere in the space of measures on ∂K and is compact in the weak-star topology. For each integer n, let $U_n = \{\lambda \in M_p |$ total variation of λ over $\partial C_0 \cup \cdots \cup \partial C_n$ exceeds $1 - \epsilon\}$. I claim that U_n is open in the weak-star topology on M_p. Once this is proved, the lemma will be finished. For $M_p = \bigcup_{n=0}^{\infty} U_n$ and hence a finite number of the U_n cover M_p. Since $U_1 \subset U_2 \subset \cdots$, there is an integer N such that $M_p = U_N$. Consequently, to establish the lemma we need only prove that U_n is open; to do this we show that $M_p - U_n$ is a weak-star closed.

Let $S = (\partial C_0 \cup \cdots \cup \partial C_n) - (F_0 \cup \cdots \cup F_n)$; S is an open subset of ∂K. Hence, if λ is a weak-star cluster point of $M_p - U_n$, then we immediately have $\|\lambda\|_S \leqq 1 - \epsilon$. However, $\lambda(F_0 \cup \cdots \cup F_n) = 0$ so that $\|\lambda\| \leqq 1 - \epsilon$ on $\partial C_0 \cup \cdots \cup \partial C_n$. Therefore, $M_p - U_n$ is weak-star closed or, equivalently, U_n is weak-star open.

THEOREM 1. *Let K be a compact subset of* **C** *and let $p \in K^0$. Let E and F be the subsets of ∂K defined above and suppose that $\lambda(E \cup F) = 0$ for all $\lambda \in M_p$. Then M_p is norm-compact.*

PROOF. Let $\{\lambda_n\}$ be a sequence in M_p; we will show that some subsequence forms a Cauchy sequence in the norm topology. Since the space of measures is complete and M_p is norm-closed, this will establish the theorem.

Let $C_0, C_1, \cdots$ be the components of $\mathbf{C} - K$ with C_0 the unbounded component. For $j = 0, 1, \cdots$ let K_j be the compact set whose complement is $C_0 \cup \cdots \cup C_j$, so that K_j has only a finite number of components in its complement and p lies in K_j^0. Let M_j be the set of representing measures on ∂K_j for evaluation at p on $R(K_j)$. For each λ in M_p and each integer $j = 0, 1, \cdots$ we define a measure $s_j\lambda$ on ∂K_j by this rule:

$$(*)\qquad \int_{\partial K_j} g\,d(s_j\lambda) = \int_{\partial K_j} g\,d\lambda + \int \tilde{g}\,d\lambda_{\partial K - \partial K_j}$$

for g in $C(\partial K_j)$ where $\tilde{g}$ is the harmonic extension of g to K_j^0. Note that $s_j\lambda$ is an element of M_j.

As the first step in the proof, we show that the total variation of $s_j\lambda - \lambda$ over ∂K may be made arbitrarily small for large values of j, simultaneously for all λ in M_p. (We consider $s_j\lambda$ to be a measure on ∂K by making it zero on $\partial K - \partial K_j$.)

If $g \in C(\partial K)$ and g_j is the restriction of g to ∂K_j, then we have by (*), $\int_{\partial K} g\,d(s_j\lambda) = \int_{\partial K_j} g\,d\lambda + \int_{\partial K - \partial K_j} \tilde{g}_j\,d\lambda$. Hence,

$$\left| \int_{\partial K} g\,d(s_j\lambda - \lambda) \right| = \left| \int_{\partial K - \partial K_j} \tilde{g}_j\,d\lambda - \int_{\partial K - \partial K_j} g\,d\lambda \right| \leqq 2\|g\|_\infty \|\lambda\|_{\partial K - \partial K_j}.$$

However, the last term is small when j is large independent of λ in M_p by Lemma 1. Consequently, the total variation of $s_j\lambda - \lambda$ over ∂K may be made arbitrarily small simultaneously for all λ in M_p by choosing j large enough.

Let us now consider the given sequence $\{\lambda_n\}$ in M_p. $\{s_1\lambda_n\}_{n=1}^\infty$ forms an infinite subset of M_1 (possibly it contains only a finite number of distinct elements of M_1; this will not affect the argument) and hence there is an element δ_1 of M_1 and a subsequence $\{s_1\lambda_{n_i}\}_{i=1}^\infty$ such that $\{s_1\lambda_{n_i}\}$ converges in norm on ∂K_1 to δ_1 as $i \to \infty$. Consider now only the subsequence $\{\lambda_{n_i}\}_{i=1}^\infty$; $\{s_2\lambda_{n_i}\}_{i=1}^\infty$ forms an infinite subset of M_2 and hence some subsequence of $\{s_2\lambda_{n_i}\}_{i=1}^\infty$ converges in norm on ∂K_2 to an element δ_2 of M_2. Some further subsequence converges in norm on ∂K_3 to an element δ_3 of M_3. Continuing this process and then extracting a diagonal sequence yields a subsequence of the original sequence, which we will again denote by $\{\lambda_n\}$, such that for each fixed j, $s_j\lambda_n$ converges in norm on ∂K_j to δ_j as $n \to \infty$.

Now we have

$$\|\lambda_n - \lambda_m\| \leqq \|\lambda_n - s_j\lambda_n\| + \|s_j\lambda_n - \delta_j\|_{\partial K_j} + \|\delta_j - s_j\lambda_m\|_{\partial K_j} + \|s_j\lambda_m - \lambda_m\|.$$

Given $\epsilon > 0$, the first and fourth terms are each smaller than $\epsilon/4$ when j is large, independent of n and m by the first part of the proof. For fixed j, the second and third terms may each be made smaller than $\epsilon/4$ by choosing n and m to be large, by the preceding paragraph of the proof. Hence, $\|\lambda_n - \lambda_m\| < \epsilon$ when n and m are large, as desired.

COROLLARY. *Let K satisfy the hypotheses of Theorem 1 and let $p \in K^0$. Let $A(K)$ be the algebra of functions continuous on K and analytic on K^0. Then the set of measures on ∂K representing evaluation at p on $A(K)$ is norm-compact.*

PROOF. These measures form a (norm)-closed subset of M_p since $R(K)$ lies inside $A(K)$. Q.E.D.

The set E in ∂K that does not lie in the boundary of any of the complementary components seems to play an important role in determining whether or not M_p is norm-compact. For example, let K be obtained by deleting from the closed unit disc a sequence of open subdiscs with disjoint closures which are centered on the positive real axis and whose centers and radii decrease to 0. The set E in this case consists of one point, the origin, and the set F is empty. If the origin is a peak point for $R(K)$ (that is, if there is a Φ in $R(K)$ with $\Phi(0)=1$ and $|\Phi(q)|<1$ for all $q\in K-\{0\}$), then Theorem 1 implies that M_p is norm-compact for each p in K^0. However, if 0 is not a peak point but is a regular point for the Dirichlet problem (as can happen; see the examples following Theorem 2), then Theorem 2 below implies that M_p is not norm-compact for any $p\in K^0$.

Before stating and proving Theorem 2, however, it will be convenient to collect in the form of a lemma some information on representing measures; all the assertions of the lemma are valid for an arbitrary function algebra, when appropriately stated.

LEMMA 2. *Let K be a compact subset of $\mathbf{C}$ and let p and q be distinct points of K which lie in the same Gleason part for $R(K)$. Then*

(i) *there is a constant c, $0<c<1$, such that $cu(p)\leqq u(q)\leqq(1/c)u(p)$ for all $u\geqq 0$, $u=\operatorname{Re}\Phi$, $\Phi\in R(K)$,*

(ii) *there is a λ in M_p with $\lambda(\{q\})>0$,*

(iii) *if M_p is norm-compact, then so is M_q.*

PROOF. The assertions of the lemma are all known with perhaps the exception of (iii); (ii) and (iii) are easy consequences of (i). Assertion (i) is proved by Bishop in [2].

THEOREM 2. *Let K be a compact subset of $\mathbf{C}$ and suppose that ∂K has zero two-dimensional Lebesque measure. Suppose that M_p is norm-compact for each $p\in K^0$. Then each point in ∂K which is regular for the Dirichlet problem is a peak point for $R(K)$ and there are at most countably many points in ∂K which are not peak points for $R(K)$.*

PROOF. If K^0 is empty, then $C(K)=R(K)$ and the conclusions are trivial; we may assume, therefore, that $K^0\neq\emptyset$. The proof for the case $K^0\neq\emptyset$ leans heavily on the following modification of a theorem of A. Browder, which appears in [3, Theorem 2]:

Let K be a compact subset of $\mathbf{C}$ of positive square Lebesque measure and suppose $p\in K$ is not a peak point for $R(K)$. Then for each $\epsilon>0$, the set of all points q in K with $\sup\{|\Phi(p)-\Phi(q)| \;|\Phi\in R(K), \|\Phi\|\leqq 1\}<\epsilon$ *has positive square Lebesque measure.*

The second assertion of Theorem 2 is the easier to prove. Since M_p is norm-compact, M_p contains a representing measure δ with the property that all the elements of M_p are absolutely continuous with respect to δ. (See [5] for the details.) This measure δ must have mass at each point in ∂K which lies in the same part for $R(K)$ as p by (ii) of Lemma 2. By Browder's Theorem and the assumption that ∂K has zero two-dimensional measure, each nonpeak point in ∂K lies in the same part for $R(K)$ as some interior component. Since there are only countably many interior components, there can be at most countably many nonpeak points in ∂K.

Browder's Theorem is also used to prove the first assertion. Suppose that $q \in \partial K$, q is regular for the Dirichlet problem, and q is not a peak point for $R(K)$. q then lies in the same part for $R(K)$ as some interior component and hence M_q, the set of representing measures on ∂K for evaluation at q on $R(K)$, is norm-compact by (iii) of Lemma 2. By Browder's Theorem there is a sequence $\{z_n\}$ of points in K^0 with $\|z_n - q\| \to 0$ where $\| \quad \|$ denotes the norm of the linear functional on $R(K)$, $\Phi \to \Phi(z_n) - \Phi(q)$. The points z_n and q are clearly in the same part for $R(K)$ and hence by (i) of Lemma 2 there is a constant c_n, $0 < c_n < 1$, such that $c_n u(q) \leqq u(z_n) \leqq (1/c_n) u(q)$ for all $u \geqq 0$, $u = \mathrm{Re}(\Phi)$, $\Phi \in R(K)$. Since $\|z_n - q\| \to 0$ an easy computation shows that we may choose the c_n so that $c_n \to 1$. For each n there is a positive measure α_n on ∂K such that $\int \Phi d\alpha_n = \Phi(q) - c_n \Phi(z_n)$; note that $1 - c_n \geqq \alpha_n(\{q\})$. Let μ_n be harmonic measure on ∂K for z_n. Then $\mu_n(\{q\}) = 0$ and, since q is regular, $\{\mu_n\}$ converges in the weak-star topology to δ_q, the point mass at q. Let $\beta_n = \alpha_n + c_n \mu_n$; then $\beta_n \in M_q$, $\beta_n(\{q\}) \leqq 1 - c_n$ and the β_n's converge in the weak-star topology to δ_q. However, $\|\beta_n - \delta_q\| \geqq 2c_n$ for each n, so that $\|\beta_n - \delta_q\| \to 2$ and hence no subsequence of the β_n can converge in norm to δ_q. Q.E.D.

Theorem 2 allows us to construct counterexamples to Bishop's conjecture. In the example cited prior to Lemma 2, if the center of the jth deleted disc is located at the point $3^{-j} + 9^{-j}$ and its radius is 9^{-j}, then the origin is regular for the Dirichlet problem while it is not a peak point for $R(K)$. See [4, p. 33] for the details.

It is possible, in fact, to construct a compact set K whose boundary has zero square Lebesgue measure and which contains a continuum of regular points, none of which is a peak point for $R(K)$. Here is one such set.

Let $\{z_j\}_{j=1}^{\infty}$ be a sequence of points within the unit disc with the property that $\mathrm{Im}\, z_j = y_j$ is positive for each j and each point of $[-\frac{1}{2}, \frac{1}{2}]$ is an accumulation point of $\bigcup_{j=1}^{\infty} \{z_j\}$ while all the accumulation points of $\bigcup_{j=1}^{\infty} \{z_j\}$ lie in $[-\frac{1}{2}, \frac{1}{2}]$. For $j = 1, 2, \cdots$ let D_j be an open disc centered at z_j of radius r_j where the r_j are chosen so small that D_j lies in the unit disc, $\overline{D}_j \cap \overline{D}_k = \emptyset$ for $j \neq k$ and $\sum_{j=1}^{\infty} r_j(y_j - r_j)^{-1} < \infty$. Let K be the closed unit disc minus $\bigcup_{j=1}^{\infty} D_j$. Then for each t in $[-\frac{1}{2}, \frac{1}{2}]$ we have

$$\int_{\partial D_j} \frac{d|z|}{|z-t|} \leqq 2\pi r_j(y_j - r_j)^{-1}$$

and hence $\int_x (d|z|/|z-t|) < \infty$ where x is the union of the unit circle and $\bigcup_{j=1}^{\infty} \partial D_j$. Cauchy's formula now extends to give

$$f(t) = \frac{1}{2\pi i} \int_x \frac{f(z)}{z-t}\, dz$$

for each f in $R(K)$ and each t in $[-\frac{1}{2}, \frac{1}{2}]$. Consequently, no point of $[-\frac{1}{2}, \frac{1}{2}]$ is a peak point for $R(K)$; however, $[-\frac{1}{2}, \frac{1}{2}]$ is a nontrivial continuum in ∂K so that each point of $[-\frac{1}{2}, \frac{1}{2}]$ is a regular for the Dirichlet problem.

Lemma 2 and Browder's Theorem hold for $A(K)$ as well as for $R(K)$ and hence the proof of Theorem 2 gives

THEOREM 3. *Let K be compact and suppose ∂K has zero square Lebesgue measure. Let $M_p(A(K))$ be those measures in M_p which represent evaluation at p on $A(K)$. If $M_p(A(K))$ is norm-compact for each p in K^0, then each regular point in ∂K for the Dirichlet problem is a peak point for $A(K)$ and there are at most countably many nonpeak points for $A(K)$ in ∂K.*

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