NORM-COMPACT SETS OF REPRESENTING MEASURES

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Let $K$ be a compact subset of the complex plane $\mathbb{C}$ and let $p$ be a point of $K^0$, the interior of $K$. Let $R(K)$ be the uniform closure on $K$ of the rational functions with poles off $K$ and let $M_p$ be the set of all positive measures $\lambda$ on $\partial K$, the topological boundary of $K$, with the property that $\int_{\partial K} \Phi d\lambda = \Phi(p)$ for all $\Phi$ in $R(K)$. (Such a measure is called a representing measure for evaluation at $p$.) In an effort to cast some light on the problem of putting analytic structure in the maximal ideal space of a function algebra, Bishop has conjectured in [1, problem 8, p. 347] that $M_p$ is compact in the norm topology as a subset of the space measures on $\partial K$. Theorem 1 of this paper shows that this is indeed the case for many compact sets; however, Theorem 2 gives some necessary conditions that $M_p$ be norm-compact and consequently provides a number of counterexamples to Bishop’s conjecture.

If the compact set has only a finite number of components in its complement, then $M_p$ is norm-compact. This follows immediately from the fact that the linear span of the real measures that annihilate $R(K)$ is finite-dimensional. (This is a consequence of a classical theorem of Walsh [6, p. 518].) However, when $K$ has infinitely many complementary components this argument is no longer valid, for both $M_p$ and the space of real annihilating measures may contain infinitely many linearly independent elements. For example, let $K$ be the set obtained by deleting from the closed unit disc a sequence $\{C_i\}$ of open subdiscs with disjoint closures whose centers lie on the positive real axis and increase to 1 and whose radii decrease to 0. Let $\mu$ be harmonic measure on $\partial K$ for $p$ and let $f_i$ be the element of $L^\infty(\partial K, \mu)$ such that for each $g$ in $L^1(\partial K, \mu)$, $\int_{\partial K} gf_i d\mu$ is the period about $\partial K$ of the harmonic conjugate of the harmonic extension of $g$ to $K^0$. It is easily seen that $f_1, f_2, \ldots$ are linearly independent and consequently that the representing measures $\lambda_i = (1 - (f_i/c_i))\mu$ are linearly independent, where $c_i = \|f_i\|_\infty$. Nevertheless, $M_p$ is norm-compact for this and other compact sets. In order to prove this we must make an assumption about two subsets of $\partial K$; these two subsets are defined below.

Let $K$ be compact and let $C_0, C_1, \cdots$ be the components of $\mathbb{C} - K$. For each integer $n$, $n = 0, 1, 2, \cdots$, let $F_n$ consist of those points of

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Fundamental to the proof of Theorem 1 is this lemma.

**Lemma 1.** Let $K$ be a compact subset of $C$ with complementary components $C_0, C_1, \ldots$, and let $E$ and $F$ be the subsets of $\partial K$ defined above. Suppose that $\lambda(E \cup F) = 0$ for all $\lambda$ in $\mathcal{M}_p$. Then given $\epsilon > 0$, there is an integer $N$ depending only on $\epsilon$ such that for each $\lambda$ in $\mathcal{M}_p$, the total variation of $\lambda$ over $\partial C_0 \cup \cdots \cup \partial C_N$ exceeds $1 - \epsilon$.

**Proof.** $\mathcal{M}_p$ is a subset of the unit sphere in the space of measures on $\partial K$ and is compact in the weak-star topology. For each integer $n$, let $U_n = \{\lambda \in \mathcal{M}_p |$ total variation of $\lambda$ over $\partial C_0 \cup \cdots \cup \partial C_n$ exceeds $1 - \epsilon\}$. I claim that $U_n$ is open in the weak-star topology on $\mathcal{M}_p$. Once this is proved, the lemma will be finished. For $\mathcal{M}_p = \bigcup_{n=0}^\infty U_n$ and hence a finite number of the $U_n$ cover $\mathcal{M}_p$. Since $U_1 \subseteq U_2 \subseteq \cdots$, there is an integer $N$ such that $\mathcal{M}_p = U_N$. Consequently, to establish the lemma we need only prove that $U_n$ is open; to do this we show that $\mathcal{M}_p - U_n$ is a weak-star closed.

Let $S = (\partial C_0 \cup \cdots \cup \partial C_n) - (F_0 \cup \cdots \cup F_n)$; $S$ is an open subset of $\partial K$. Hence, if $\lambda$ is a weak-star cluster point of $\mathcal{M}_p - U_n$, then we immediately have $||\lambda||_S \leq 1 - \epsilon$. However, $\lambda(F_0 \cup \cdots \cup F_n) = 0$ so that $||\lambda|| \leq 1 - \epsilon$ on $\partial C_0 \cup \cdots \cup \partial C_n$. Therefore, $\mathcal{M}_p - U_n$ is weak-star closed or, equivalently, $U_n$ is weak-star open.

**Theorem 1.** Let $K$ be a compact subset of $C$ and let $p \in K^0$. Let $E$ and $F$ be the subsets of $\partial K$ defined above and suppose that $\lambda(E \cup F) = 0$ for all $\lambda \in \mathcal{M}_p$. Then $\mathcal{M}_p$ is norm-compact.

**Proof.** Let $\{\lambda_n\}$ be a sequence in $\mathcal{M}_p$; we will show that some subsequence forms a Cauchy sequence in the norm topology. Since the space of measures is complete and $\mathcal{M}_p$ is norm-closed, this will establish the theorem.

Let $C_0, C_1, \cdots$ be the components of $C - K$ with $C_0$ the unbounded component. For $j = 0, 1, \cdots$ let $K_j$ be the compact set whose complement is $C_0 \cup \cdots \cup C_j$, so that $K_j$ has only a finite number of components in its complement and $p$ lies in $K_j^0$. Let $M_j$ be the set of representing measures on $\partial K_j$ for evaluation at $p$ on $R(K_j)$. For each $\lambda$ in $\mathcal{M}_p$ and each integer $j = 0, 1, \cdots$ we define a measure $s_j \lambda$ on $\partial K_j$ by this rule:
for $g$ in $C(\partial K_j)$ where $\tilde{g}$ is the harmonic extension of $g$ to $K_j^0$. Note that $s_j\lambda$ is an element of $M_j$.

As the first step in the proof, we show that the total variation of $s_j\lambda - \lambda$ over $\partial K$ may be made arbitrarily small for large values of $j$, simultaneously for all $\lambda$ in $M_p$. (We consider $s_j\lambda$ to be a measure on $\partial K$ by making it zero on $\partial K - \partial K_j$.)

If $g \in C(\partial K)$ and $g_j$ is the restriction of $g$ to $\partial K_j$, then we have by (*), $\int_{\partial K} gd(s_j\lambda) = \int_{\partial K} g d\lambda + \int_{\partial K - \partial K_j} \tilde{g} d\lambda$. Hence,

$$\left| \int_{\partial K} gd(s_j\lambda - \lambda) \right| = \left| \int_{\partial K} \tilde{g} d\lambda - \int_{\partial K - \partial K_j} \tilde{g} d\lambda \right| \leq 2\|g\|_{\infty} \|\lambda\|_{\partial K - \partial K_j}.$$

However, the last term is small when $j$ is large independent of $\lambda$ in $M_p$ by Lemma 1. Consequently, the total variation of $s_j\lambda - \lambda$ over $\partial K$ may be made arbitrarily small simultaneously for all $\lambda$ in $M_p$ by choosing $j$ large enough.

Let us now consider the given sequence $\{\lambda_n\}$ in $M_p$. $\{s_i\lambda_n\}_{n=1}^{\infty}$ forms an infinite subset of $M_1$ (possibly it contains only a finite number of distinct elements of $M_1$; this will not affect the argument) and hence there is an element $\delta_1$ of $M_1$ and a subsequence $\{s_1\lambda_{n_i}\}_{i=1}^{\infty}$ such that $\{s_1\lambda_{n_i}\}$ converges in norm on $\partial K_1$ to $\delta_1$ as $i \to \infty$. Consider now only the subsequence $\{\lambda_{n_i}\}_{i=1}^{\infty}; \{s_2\lambda_{n_i}\}_{i=1}^{\infty}$ forms an infinite subset of $M_2$ and hence some subsequence of $\{s_2\lambda_{n_i}\}_{i=1}^{\infty}$ converges in norm on $\partial K_2$ to an element $\delta_2$ of $M_2$. Some further subsequence converges in norm on $\partial K_3$ to an element $\delta_3$ of $M_3$. Continuing this process and then extracting a diagonal sequence yields a subsequence of the original sequence, which we will again denote by $\{\lambda_n\}$, such that for each fixed $j$, $s_j\lambda_n$ converges in norm on $\partial K_j$ to $\delta_j$ as $n \to \infty$.

Now we have

$$\|\lambda_n - \lambda_m\| \leq \|\lambda_n - s_j\lambda_n\| + \|s_j\lambda_n - \delta_j\|_{\partial K_j} + \|\delta_j - s_j\lambda_m\|_{\partial K_j} + \|s_j\lambda_m - \lambda_m\|.$$

Given $\epsilon > 0$, the first and fourth terms are each smaller than $\epsilon/4$ when $j$ is large, independent of $n$ and $m$ by the first part of the proof. For fixed $j$, the second and third terms may each be made smaller than $\epsilon/4$ by choosing $n$ and $m$ to be large, by the preceding paragraph of the proof. Hence, $\|\lambda_n - \lambda_m\| < \epsilon$ when $n$ and $m$ are large, as desired.

**Corollary.** Let $K$ satisfy the hypotheses of Theorem 1 and let $p \in K^0$. Let $A(K)$ be the algebra of functions continuous on $K$ and analytic on $K^0$. Then the set of measures on $\partial K$ representing evaluation at $p$ on $A(K)$ is norm-compact.
These measures form a (norm)-closed subset of $M_p$ since $R(K)$ lies inside $A(K)$. Q.E.D.

The set $E$ in $\partial K$ that does not lie in the boundary of any of the complementary components seems to play an important role in determining whether or not $M_p$ is norm-compact. For example, let $K$ be obtained by deleting from the closed unit disc a sequence of open subdiscs with disjoint closures which are centered on the positive real axis and whose centers and radii decrease to 0. The set $E$ in this case consists of one point, the origin, and the set $F$ is empty. If the origin is a peak point for $R(K)$ (that is, if there is a $\Phi$ in $R(K)$ with $\Phi(0) = 1$ and $|\Phi(q)| < 1$ for all $q \in K - \{0\}$), then Theorem 1 implies that $M_p$ is norm-compact for each $p$ in $K^0$. However, if 0 is not a peak point but is a regular point for the Dirichlet problem (as can happen; see the examples following Theorem 2), then Theorem 2 below implies that $M_p$ is not norm-compact for any $p \in K^0$.

Before stating and proving Theorem 2, however, it will be convenient to collect in the form of a lemma some information on representing measures; all the assertions of the lemma are valid for an arbitrary function algebra, when appropriately stated.

**Lemma 2.** Let $K$ be a compact subset of $\mathcal{C}$ and let $p$ and $q$ be distinct points of $K$ which lie in the same Gleason part for $R(K)$. Then

(i) there is a constant $c$, $0 < c < 1$, such that $c u(p) \leq u(q) \leq (1/c) u(p)$ for all $u \geq 0$, $u = \text{Re} \Phi, \Phi \in R(K)$,

(ii) there is a $\lambda$ in $M_p$ with $\lambda(\{q\}) > 0$,

(iii) if $M_p$ is norm-compact, then so is $M_q$.

**Proof.** The assertions of the lemma are all known with perhaps the exception of (iii); (ii) and (iii) are easy consequences of (i). Assertion (i) is proved by Bishop in [2].

**Theorem 2.** Let $K$ be a compact subset of $\mathcal{C}$ and suppose that $\partial K$ has zero two-dimensional Lebesque measure. Suppose that $M_p$ is norm-compact for each $p \in K^0$. Then each point in $\partial K$ which is regular for the Dirichlet problem is a peak point for $R(K)$ and there are at most countably many points in $\partial K$ which are not peak points for $R(K)$.

**Proof.** If $K^0$ is empty, then $C(K) = R(K)$ and the conclusions are trivial; we may assume, therefore, that $K^0 \neq \emptyset$. The proof for the case $K^0 \neq \emptyset$ leans heavily on the following modification of a theorem of A. Browder, which appears in [3, Theorem 2]:

Let $K$ be a compact subset of $\mathcal{C}$ of positive square Lebesque measure and suppose $p \in K$ is not a peak point for $R(K)$. Then for each $\epsilon > 0$, the set of all points $q$ in $K$ with $\{ |\Phi(p) - \Phi(q)| : \Phi \in R(K), ||\Phi|| \leq 1 \} < \epsilon$ has positive square Lebesque measure.
The second assertion of Theorem 2 is the easier to prove. Since $M_p$ is norm-compact, $M_p$ contains a representing measure $\delta$ with the property that all the elements of $M_p$ are absolutely continuous with respect to $\delta$. (See [5] for the details.) This measure $\delta$ must have mass at each point in $\partial K$ which lies in the same part for $R(K)$ as $p$ by (ii) of Lemma 2. By Browder's Theorem and the assumption that $\partial K$ has zero two-dimensional measure, each nonpeak point in $\partial K$ lies in the same part for $R(K)$ as some interior component. Since there are only countably many interior components, there can be at most countably many nonpeak points in $\partial K$.

Browder's Theorem is also used to prove the first assertion. Suppose that $q \in \partial K$, $q$ is regular for the Dirichlet problem, and $q$ is not a peak point for $R(K)$. $q$ then lies in the same part for $R(K)$ as some interior component and hence $M_q$, the set of representing measures on $\partial K$ for evaluation at $q$ on $R(K)$, is norm-compact by (iii) of Lemma 2. By Browder's Theorem there is a sequence $\{z_n\}$ of points in $K^0$ with $\|z_n - q\| \to 0$ where $\|\|$ denotes the norm of the linear functional on $R(K)$, $\Phi \to \Phi(z_n) - \Phi(q)$. The points $z_n$ and $q$ are clearly in the same part for $R(K)$ and hence by (i) of Lemma 2 there is a constant $c_n$, $0 < c_n < 1$, such that $c_n u(q) \leq u(z_n) \leq (1/c_n) u(q)$ for all $u \geq 0$, $u = \text{Re}(\Phi), \Phi \in R(K)$. Since $\|z_n - q\| \to 0$ an easy computation shows that we may choose the $c_n$ so that $c_n \to 1$. For each $n$ there is a positive measure $\alpha_n$ on $\partial K$ such that $\int \Phi d\alpha_n = \Phi(q) - c_n \Phi(z_n)$; note that $1 - c_n \geq \alpha_n(\{q\})$. Let $\mu_n$ be harmonic measure on $\partial K$ for $z_n$. Then $\mu_n(\{q\}) = 0$ and, since $q$ is regular, $\{\mu_n\}$ converges in the weak-star topology to $\delta_q$, the point mass at $q$. Let $\beta_n = \alpha_n + c_n \mu_n$; then $\beta_n \in M_q$, $\beta_n(\{q\}) \leq 1 - c_n$ and the $\beta_n$'s converge in the weak-star topology to $\delta_q$. However, $\|\beta_n - \delta_q\| \geq 2c_n$ for each $n$, so that $\|\beta_n - \delta_q\| \to 2$ and hence no subsequence of the $\beta_n$ can converge in norm to $\delta_q$. Q.E.D.

Theorem 2 allows us to construct counterexamples to Bishop's conjecture. In the example cited prior to Lemma 2, if the center of the $j$th deleted disc is located at the point $3^{-j} + 9^{-j}$ and its radius is $9^{-j}$, then the origin is regular for the Dirichlet problem while it is not a peak point for $R(K)$. See [4, p. 33] for the details.

It is possible, in fact, to construct a compact set $K$ whose boundary has zero square Lebesgue measure and which contains a continuum of regular points, none of which is a peak point for $R(K)$. Here is one such set.

Let $\{z_j\}_{j=1}^\infty$ be a sequence of points within the unit disc with the property that $\text{Im } z_j = y_j$ is positive for each $j$ and each point of $[-\frac{1}{2}, \frac{1}{2}]$ is an accumulation point of $U_j = \{z_j\}$ while all the accumulation points of $U_j$ lie in $[-\frac{1}{2}, \frac{1}{2}]$. For $j = 1, 2, \cdots$ let $D_j$ be an open disc centered at $z_j$ of radius $r_j$ where the $r_j$ are chosen so small
that \( D_j \) lies in the unit disc, \( \overline{D}_j \cap \overline{D}_k = \emptyset \) for \( j \neq k \) and \( \sum_{j=1}^{n} r_j(y_j - r_j)^{-1} < \infty \). Let \( K \) be the closed unit disc minus \( \bigcup_{j=1}^{n} D_j \). Then for each \( t \) in \( [-\frac{1}{2}, \frac{1}{2}] \) we have

\[
\int_{\partial D_j} \frac{d|z|}{|z-t|} \leq 2\pi r_j(y_j - r_j)^{-1}
\]

and hence \( f_x (d|z|/|z-t|) < \infty \) where \( x \) is the union of the unit circle and \( \bigcup_{j=1}^{n} \partial D_j \). Cauchy's formula now extends to give

\[
f(t) = \frac{1}{2\pi i} \int_{x} \frac{f(z)}{z-t} \, dz
\]

for each \( f \) in \( R(K) \) and each \( t \) in \( [-\frac{1}{2}, \frac{1}{2}] \). Consequently, no point of \( [-\frac{1}{2}, \frac{1}{2}] \) is a peak point for \( R(K) \); however, \( [-\frac{1}{2}, \frac{1}{2}] \) is a nontrivial continuum in \( \partial K \) so that each point of \( [-\frac{1}{2}, \frac{1}{2}] \) is a regular for the Dirichlet problem.

Lemma 2 and Browder's Theorem hold for \( A(K) \) as well as for \( R(K) \) and hence the proof of Theorem 2 gives

**Theorem 3.** Let \( K \) be compact and suppose \( \partial K \) has zero square Lebesgue measure. Let \( M_p(A(K)) \) be those measures in \( M_p \) which represent evaluation at \( p \) on \( A(K) \). If \( M_p(A(K)) \) is norm-compact for each \( p \) in \( K^0 \), then each regular point in \( \partial K \) for the Dirichlet problem is a peak point for \( A(K) \) and there are at most countably many nonpeak points for \( A(K) \) in \( \partial K \).

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**References**


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